Antidifferentiation by Substitution

Chapter 7 gives us some techniques for finding integrals or antiderivatives.

Definition: Indefinite Integral

The family of all antiderivatives of a function f(x) is the **indefinite integral of *f* with respect to x** and is denoted by $\int f(x)\,dx$. If F is any function such that $F'(x) = f(x), the \int f(x)dx = F(x) + C$, where C is an arbitrary constant, called the **constant of integration**.

$\int$ is the **integral sign**, the function *f* is the **integrand** of the integral and x is the **variable of integration**

When we find F(x) + C we have **integrated** f or **evaluated** the integral.

Example 1

Properties of Indefinite Integrals

$\int k\,f(x)dx = k\int f(x)dx$ for any constant k

$$\int (f(x) \pm g(x))\,dx = \int f(x)dx \pm \int g(x)dx$$

Power Formulas

$\int u^n du = \frac{u^{n+1}}{n+1} + C, when\ n \neq -1$

$\int u^{-1} du = \int \frac{1}{u} du = ln|u| + C$

Trigonometric Formulas

$\int \cos u\,du = \sin u + C$

$\int \sin u\,du = -\cos u + C$

$\int sec^2\,u\,du = \tan u + C$

$\int csc^2\,u\,du = -\cot u + C$

$\int \sec(u)\tan(u)\,du = \sec u + C$

$\int \csc(u)\cot(u)\,du = -\csc u + C$

Exponential and Logarithmic Formulas

$\int e^u du = e^u + C$

$\int a^u du = \frac{a^u}{\ln a} + C$

$$\int \ln u\,du = u \ln u - u + C$$

$$\int \log_a u\,du = \int \frac{\ln u}{\ln a} du = \frac{u\ln(u) - u}{\ln a} + C$$

Example 2 verifies Antiderivative formulas

Example 3 Paying Attention to the Differential

Here is a table that might be helpful in antidifferentiating functions

Indefinite Integral	Reversed Derivative Formula
$\int x^n dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$	$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n,\ n \neq -1$
$\int \frac{1}{x} dx = \ln\lvert x\rvert + C$	$\frac{d}{dx}\ln\lvert x\rvert = \frac{1}{x}$
$\int e^{kx} dx = \frac{e^{kx}}{k} + C$	$\frac{d}{dx}\frac{e^{kx}}{k} = e^{kx}$
$\int \sin kx\ dx = -\frac{\cos kx}{k} + C$	$\frac{d}{dx}\left(-\frac{\cos kx}{k}\right) = \sin kx$
$\int \cos kx\ dx = \frac{\sin kx}{k} + C$	$\frac{d}{dx}\left(\frac{\sin kx}{k}\right) = \cos kx$
$\int \sec^2 x\ dx = \tan x + C$	$\frac{d}{dx}\tan x = \sec^2 x$
$\int \csc^2 x\ dx = -\cot x + C$	$\frac{d}{dx}(-\cot x) = \csc^2 x$
$\int \sec x \tan x\ dx = \sec x + C$	$\frac{d}{dx}\sec x = \sec x \tan x$
$\int \csc x \cot x\ dx = -\csc x + C$	$\frac{d}{dx}(-\csc x) = \csc x \cot x$

Remember $\int x^n\ dx = \frac{x^{n+1}}{n+1} + C$

$f \qquad\qquad F$

Example: $\int x^2\ dx \quad = \quad \frac{x^3}{3} + C$

Recall also the power for derivatives could be used even if the base was a function by using the chain rule

$$\frac{d}{dx}\frac{(2x+1)^5}{5} = \frac{5(2x+1)^4 \cdot 2}{5}$$

$$\frac{d}{dx}\frac{(u)^{n+1}}{n+1} = \frac{u^n \cdot du}{dx}$$

The inverse of the power rule and chain rule together is the power rule for integrals.

$$\int u^n du = \frac{u^{n+1}}{n+1} + C \qquad n \neq -1 \qquad if\ \ n = -1 \qquad then\ \ u^n = \frac{1}{u} \quad \Rightarrow \quad \int \frac{1}{u} du = \ln u + C$$

This formula is very helpful, but often hard to recognize.

Example: $\int \left(x^3-2\right)^4 3x^2\, dx$ To find this we need an antiderivative of the integrand. The integrand is a product. This is difficult to integrate. However, if we integrate using a "u" substitution this becomes easy.

$$\int \left(x^3-2\right)^4 3x^2\, dx$$

$$Let\, u = x^3-2\, and\ du = 3x^2dx \Rightarrow \int u^4 du \Rightarrow \frac{u^5}{5} \Rightarrow \frac{\left(x^3-2\right)^5}{5} + C$$

Check the answer by finding the derivative.

Do these problems:

$$\int \left(x+2\right)^3 dx$$

$$\int \sqrt{4x-1}\, dx$$

We do a "u" substitution when a function and its derivative (or a constant multiple of the derivative) make up the integral.

#20 $\int 28\left(7x-2\right)^3 dx$

#17 $\int \sin 3x\, dx$

From #17 we can see that some trig functions aren't too bad to integrate.

Example: $\int \sec^2 x\ dx$

Example 4

Example 5

What about $\int \tan x \, dx$

Since tan(x) isn't in our integral table and it's not the derivative of any function we know, we'll have to do something different. Remember that all functions can be written in terms of sines and cosines. Try changing tangent to sin/cos

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx$$

What do you notice? Answer: **sin and cos are derivatives of one another**

So we should do a "u" substitution. But, which one is u?

$$\begin{array}{lll} u = \sin x & or & u = \cos x \\ du = \cos x \, dx & & du = -\sin x \, dx \\ doesn't\ work & & \int \frac{-1}{u} du \end{array}$$

$$\int \frac{-1}{u} du = -\ln u + C = -\ln\left|\cos x\right| + C$$

$$or \quad \ln\left|\frac{1}{\cos x}\right| + C \qquad or \qquad \ln\left|\sec x\right| + C$$

Substitution in Indefinite Integrals

$\int f(g(x)) \cdot g'(x) \ dx \ = \ \int f(u) \, du$	Substitute u=g(x), du=g′ (x) dx
= F(u) + C	Evaluate by finding an antiderivative F(u) of f (u)
=F(g(x)) + C	Replace u by g(x)

Example 7:

Substitution in Definite Integrals
Substitute u = g (x), du = g′ (x) dx   and integrate with respect to u from u = g (a) to u = g (b)

$$\int_a^b f(g(x)) \cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Example 8 $\int_0^{\frac{\pi}{4}} \tan x \sec^2 x\,dx$ What makes this problem different?

Continue on as normal with u = tan x and du = $\sec^2$ x dx

$$\int_0^{\frac{\pi}{4}} u\,du = \left[\frac{u^2}{2}\right]_0^{\frac{\pi}{4}} = \frac{\left(\tan\frac{\pi}{4}\right)^2}{2} - \frac{(\tan 0)^2}{2} = \frac{1}{2} - 0 = \frac{1}{2}$$

Or change the limits of integration by evaluating u at the limits.

$$\int_{\tan 0}^{\tan\frac{\pi}{4}} u\,du = \int_0^1 u\,du = \left[\frac{u^2}{2}\right]_0^1 = \frac{1}{2} - 0 = \frac{1}{2}$$

It’s the same either way. You just have to decide if evaluating “u” at the limits saves effort. Usually it does.

Example 9