Vectors

Scalar Quantities: Quantities such as length, area, volume, temperature, and time, which have magnitude (size), but no direction.

Vector Quantities: Quantities that involve both a magnitude and direction, such as velocity, acceleration, and force. These quantities can be represented by **directed line segments** called **vectors**.

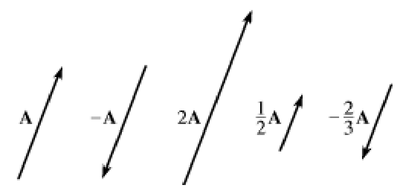
The length of a vector represents the **magnitude** of the vector quantity. The **direction** is indicated by the position of the vector and the arrowhead at one end.

Notation: $\overrightarrow{AB}$ is used to name a vector with **initial point** A and **terminal point** B. Vectors may also be denoted by bold letters. $\overrightarrow{AB}$ can also be written as $\mathbf{AB}$. If the initial and terminal points are not specified, vectors can be named by a single uppercase or lowercase letter (eg. $\vec{b}$, $\vec{B}$, **b**, or **B**). The magnitude of vector $\mathbf{A}$ is written $|\mathbf{A}|$.

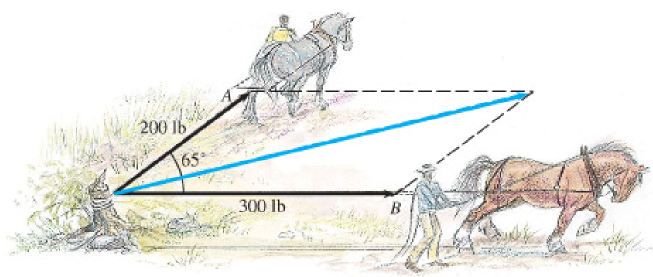
Equal Vectors: Vectors with the same magnitude and direction. They do not have to be in the same place.

Zero Vector: A vector with no magnitude and no direction. It is denoted by $\mathbf{0}$.

Scalar Multiplication: For any scalar k and vector **A,** $k\mathbf{A}$ is a vector with magnitude $|k|$ times the magnitude of **A**. If $k > 0$, then the direction of $k\mathbf{A}$ is the same as the direction of **A**. If $k < 0$, the direction of $k\mathbf{A}$ is opposite to the direction of **A**. If $k = 0$, then $k\mathbf{A} = \mathbf{0}$.

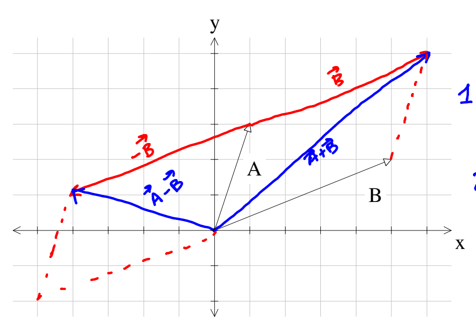


Suppose two draft horses are pulling on a tree stump with forces of 200 pounds and 300 pounds, with an angle of 65° between the forces. If **A** and **B** had the same direction, then there would be a total force of 500 pounds acting on the stump, but the total force is less because of the angle between the forces. By the **parallelogram law**, the force acting along the diagonal of the parallelogram, with a magnitude equal to the length of the diagonal, has the same effect on the stump as the two forces **A** and **B**. The force **A + B** acting along the diagonal is called the **sum** or **resultant** of **A** and **B**.



Vector Addition: To find the resultant or sum **A + B** of any vectors **A** and **B**, draw **B** so that the initial point of **B** is at the terminal point of **A**. The vector that begins at the initial point of **A** and ends at the terminal point of **B** is the vector **A + B**. For every vector **B**, there is a vector **– B**, with the same magnitude as **B**, but the opposite direction. **A – B = A + (– B)**.

Example: Sketch the vectors **A + B** and **A – B**.



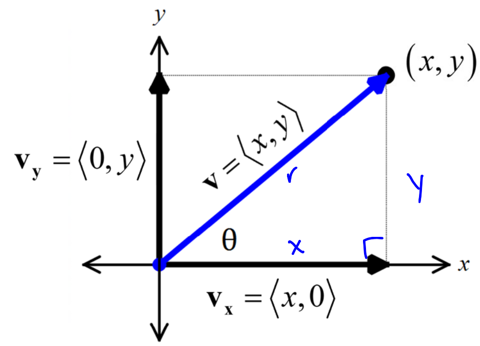
(or $-\vec{B}$)

1. Draw $\vec{B}$ from end of $\vec{A}$.

2. Connect beginning of $\vec{A}$ to end of $\vec{B}$.

(or $-\vec{B}$)

Any nonzero vector **v** is the sum of a **horizontal component**, $\mathbf{v}_x$, and a **vertical component**, $\mathbf{v}_y$. If a vector **v** is placed in a rectangular coordinate system so that its initial point is the origin, then **v** is called a **position vector**. The angle θ formed by the positive x-axis and a position vector is the **direction angle** for the position vector.



Component Form: The notation $\langle x, y\rangle$ is used to define a position vector with terminal point (x, y). This is called component form because the horizontal component is $\mathbf{v}_x = \langle x, 0\rangle$ and its vertical component is $\mathbf{v}_y = \langle 0, y\rangle$.

The magnitude of the vector $\mathbf{v} = \langle x, y\rangle$ is $|\mathbf{v}| = r = \sqrt{x^2 + y^2}$. To find the direction angle, use

$\sin\theta = \dfrac{y}{r}$, $\cos\theta = \dfrac{x}{r}$, and $\tan\theta = \dfrac{y}{x}$.

If a vector has magnitude r and direction angle θ, its component form is $\langle r\cos\theta,\ r\sin\theta\rangle$.

Examples: Find the component form for each vector **v** with the given magnitude and direction angle θ. Round to the nearest tenth if necessary.

$\langle r\cos\theta, r\sin\theta\rangle$

a) $|\mathbf{v}| = 12,\ \theta = 45°$

$\cos 45° = \dfrac{x}{12}$

$x = 12\cos 45°$
$= 12\left(\dfrac{\sqrt{2}}{2}\right) = 6\sqrt{2}$

$\sin 45° = \dfrac{y}{12}$

$y = 12\sin 45°$
$= 12\left(\dfrac{\sqrt{2}}{2}\right) = 6\sqrt{2}$

$\boxed{\langle 6\sqrt{2}, 6\sqrt{2}\rangle}$

b) $|\mathbf{v}| = 50,\ \theta = 120°$

$\langle 50\cos 120°, 50\sin 120°\rangle$

$\left\langle 50\left(-\tfrac{1}{2}\right), 50\left(\tfrac{\sqrt{3}}{2}\right)\right\rangle$

$\boxed{\langle -25, 25\sqrt{3}\rangle}$

c) $|\mathbf{v}| = 445,\ \theta = 211.1°$

$\langle 445\cos 211.1°, 445\sin 211.1°\rangle$

$\boxed{\langle -381.0, -229.9\rangle}$

Examples: Find the magnitude and direction angle of each vector.

a) $\mathbf{v} = \langle 1, -1\rangle$

$r = |\langle 1,-1\rangle| = \sqrt{1^2+(-1)^2} = \boxed{\sqrt{2}}$

$\sin\theta = \dfrac{-1}{\sqrt{2}} = -\dfrac{\sqrt{2}}{2}$

$\boxed{\theta = 315°}$

could also use cosine or tangent, or realize it's a 45°-45°-90° triangle. Doesn't matter!

b) $\mathbf{v} = \langle -2,\ 2\sqrt{3}\rangle$

30°-60°-90°
hyp = 2·short leg

$r = 2(2)$
$r = 4$

$\theta = 180° - 60°$
$\theta = 120°$

fast & easy

$\boxed{\begin{array}{l} r = 4 \\ \theta = 120° \end{array}}$

$r = \sqrt{2^2 + (2\sqrt{3})^2}$
$= \sqrt{4 + (4\cdot 3)} = \sqrt{4+12}$
$= \sqrt{16} = 4$

$\cos\theta = \dfrac{-2}{4} = -\dfrac{1}{2}$

could also use sine or tangent $\theta = 120°$

way longer!

c) $\mathbf{v} = \langle -4, -5\rangle$

$r = \sqrt{4^2 + 5^2} = \boxed{\sqrt{41}}$

tangent is easiest for non-unit circle angles because you can add 180° if you get something in the wrong quadrant.

$\tan\theta = \dfrac{-5}{-4} = \dfrac{5}{4}$ $\quad \tan^{-1}\left(\dfrac{5}{4}\right) = 51.3°$

$\theta \approx 51.3° + 180° \approx \boxed{231.3°}$

If $\mathbf{A} = \langle a_1, a_2\rangle$, $\mathbf{B} = \langle b_1, b_2\rangle$, and k is a scalar, then

Vector Arithmetic:

1.	$k\mathbf{A} = \langle ka_1, ka_2\rangle$	**Scalar Product**
2.	$\mathbf{A} + \mathbf{B} = \langle a_1 + b_1, a_2 + b_2\rangle$	**Vector Sum**
3.	$\mathbf{A} - \mathbf{B} = \langle a_1 - b_1, a_2 - b_2\rangle$	**Vector Difference**
4.	$\mathbf{A}\cdot\mathbf{B} = a_1 b_1 + a_2 b_2$	**Dot Product**

a dot product is a number, not a vector!!!

Examples: Let $\mathbf{w} = \langle -1,-3 \rangle$ and $\mathbf{v} = \langle -3,4 \rangle$. Perform the operations indicated.

a) $\mathbf{w} - \mathbf{v}$

$\langle -1,-3 \rangle - \langle -3,4 \rangle$

$= \boxed{\langle 2,-7 \rangle}$

b) $-8\mathbf{v}$

$-8\langle -3,4 \rangle$

$= \boxed{\langle 24,-32 \rangle}$

c) $3\mathbf{w} + 4\mathbf{v}$

$3\langle -1,-3 \rangle + 4\langle -3,4 \rangle$

$= \langle -3,-9 \rangle + \langle -12,16 \rangle$

$= \boxed{\langle -15,7 \rangle}$

d) $\mathbf{w} \cdot \mathbf{v}$

$\langle -1,-3 \rangle \cdot \langle -3,4 \rangle$

$= 3 - 12$ ← Don't put in $\langle\ \rangle$

$= \boxed{-9}$

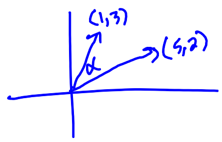
↑ #, not vector!!!

The Angle Between Two Vectors:

If **A** and **B** are nonzero vectors and α is the smallest positive angle between them, then $\alpha = \cos^{-1}\left(\dfrac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}||\mathbf{B}|}\right)$.

Examples: Find the smallest positive angle between the following vectors:

a) $\langle 1,3 \rangle$ and $\langle 5,2 \rangle$



$\langle 1,3 \rangle \cdot \langle 5,2 \rangle$

$5 + 6 = 11$

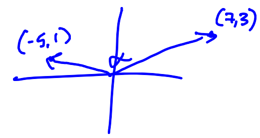
$|\langle 1,3 \rangle| = \sqrt{1^2+3^2} = \sqrt{10}$

$|\langle 5,2 \rangle| = \sqrt{5^2+2^2} = \sqrt{29}$

$\alpha = \cos^{-1}\left(\dfrac{11}{\sqrt{10}\cdot\sqrt{29}}\right)$

$\boxed{\alpha \approx 49.8^\circ}$

b) $\langle -5,1 \rangle$ and $\langle 7,3 \rangle$



$\langle -5,1 \rangle \cdot \langle 7,3 \rangle$

$= -35 + 3 = -32$

$|\langle -5,1 \rangle| = \sqrt{5^2+1^2} = \sqrt{26}$

$|\langle 7,3 \rangle| = \sqrt{7^2+3^2} = \sqrt{58}$

$\alpha = \cos^{-1}\left(\dfrac{-32}{\sqrt{26}\cdot\sqrt{58}}\right) \approx \boxed{145.5^\circ}$

← same slope → y/x is same

Parallel Vectors: The vectors **A** and **B** are parallel if and only if $\mathbf{A} = k\mathbf{B}$ for a nonzero scalar k.
Perpendicular Vectors: The vectors **A** and **B** are perpendicular if and only if $\mathbf{A}\cdot\mathbf{B} = 0$. or slopes are negative reciprocals

Examples: Determine whether each pair of vectors is parallel, perpendicular, or neither.

a) $\langle -2,3 \rangle$ and $\langle 6,4 \rangle$

Not parallel

$\frac{3}{-2} \neq \frac{4}{6}$

$\langle -2,3 \rangle \cdot \langle 6,4 \rangle$

$= -12 + 12 = 0$

$\boxed{\text{perpendicular}}$

slopes: $-3/2$ & $2/3$ ← negative reciprocals

b) $\langle 2,-5 \rangle$ and $\langle -4,10 \rangle$

$\boxed{\text{parallel}}$

$-2\langle 2,-5 \rangle$

$= \langle -4,10 \rangle$

or $\frac{-5}{2} = \frac{10}{-4}$

↑ slopes equal

c) $\langle 2,6 \rangle$ and $\langle 6,2 \rangle$

not parallel

$\frac{6}{2} \neq \frac{2}{6}$

$\boxed{\text{neither}}$

not negative reciprocals

not perpendicular ←

$\langle 2,6 \rangle \cdot \langle 6,2 \rangle = 12 + 12 = 24$

The vectors $\mathbf{i} = \langle 1,0 \rangle$ and $\mathbf{j} = \langle 0,1 \rangle$ are called **unit vectors** because each has magnitude one. For any vector $\langle a_1, a_2 \rangle$, we have $\langle a_1, a_2 \rangle = a_1\langle 1,0 \rangle + a_2\langle 0,1 \rangle = a_1\mathbf{i} + a_2\mathbf{j}$. The form $a_1\mathbf{i} + a_2\mathbf{j}$ is called a **linear combination** of the vectors **i** and **j**.

Examples: Write each vector as a linear combination of the unit vectors **i** and **j**.

a) $\mathbf{A} = \langle 2,3 \rangle$

$\vec{A} = 2\vec{i} + 3\vec{j}$

b) $\mathbf{B} = \langle -1,7 \rangle$

$\vec{B} = -\vec{i} + 7\vec{j}$

c) $\mathbf{C} = \langle 0,-9 \rangle$

$\vec{C} = -9\vec{j}$