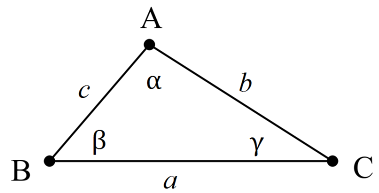
The Law of Cosines



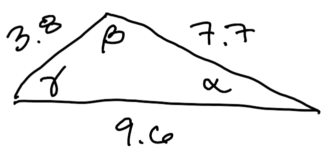
The Law of Cosines: In any triangle, $a^2 = b^2 + c^2 - 2bc\cos\alpha$

$$b^2 = a^2 + c^2 - 2ac\cos\beta$$

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

SSS: Use the fact that the largest angle is across from the longest side of the triangle to solve for the largest angle using the law of cosines. (For example, if c is the longest side, use the equation $c^2 = a^2 + b^2 - 2ab\cos\gamma$ to solve for γ.) Then use the law of sines to find the remaining angles, which will both be acute. **Don't use the law of sines to solve for any angle that might be obtuse! The law of sines will always give you acute angle measures!**

Example: $a = 3.8,\ b = 9.6,\ c = 7.7$



$9.6^2 = 3.8^2 + 7.7^2 - 2(3.8)(7.7)\cos\beta$

$92.16 = 73.73 - 58.52\cos\beta$

$-73.73 \quad -73.73$

$\frac{18.43}{-58.52} = \frac{-58.52\cos\beta}{-58.52}$

$\cos\beta \approx -.3149$

$\boxed{\beta \approx 108.4^\circ}$

$\frac{\sin 108.4^\circ}{9.6} = \frac{\sin\gamma}{7.7}$

$\sin\gamma = \frac{7.7\sin 108.4^\circ}{9.6} \approx .7613$

$\boxed{\gamma \approx 49.6^\circ}$

$\alpha = 180^\circ - (108.4^\circ + 49.6^\circ)$

$\boxed{\alpha \approx 22.1^\circ}$

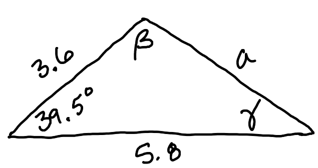
SAS: Find the length of the third side using the law of cosines. Use the law of sines to find the angle across from the shorter of the two given sides. Find the remaining angle by subtracting the first two from 180°.

Example: $b = 5.8,\ c = 3.6,\ \alpha = 39.5^\circ$

$a^2 = 3.6^2 + 5.8^2 - 2(3.6)(5.8)\cos 39.5^\circ$

$a^2 \approx 14.377$

$\boxed{a \approx 3.8}$



Warning: Since β might obtuse (β is the biggest angle, since it's across from the longest side), don't use Law of Sines to find β. Law of Sines always gives acute angles!

$\frac{\sin 39.5^\circ}{3.8} = \frac{\sin\gamma}{3.6}$

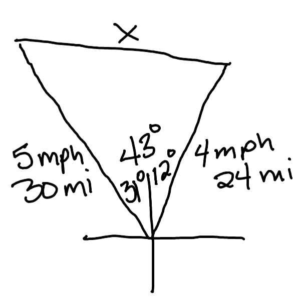
$\sin\gamma = \frac{3.6\sin 39.5^\circ}{3.8} \approx .6039$

$\boxed{\gamma = 37.2^\circ}$

$\beta = 180^\circ - (39.5^\circ + 37.2^\circ)$

$\boxed{\beta \approx 103.3^\circ}$

Example: Jan and Dean started hiking from the same location at the same time. Jan hiked at 4 mph with bearing N12°E, and Dean hiked at 5 mph with bearing N31°W. How far apart were they after 6 hours? Round to the nearest tenth of a mile.

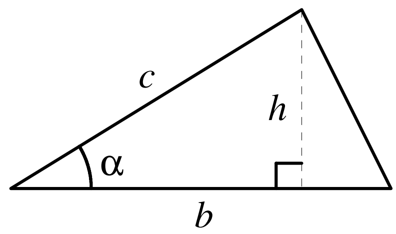


$x^2 = 30^2 + 24^2 - 2(30)(24)\cos 43^\circ$

$x^2 \approx 422.85$

$\boxed{x \approx 20.6\text{ mi}}$

Area of a Triangle



Area of triangle = $\frac{1}{2}bh$

$\sin\alpha = \frac{h}{c}$

$h = c\sin\alpha$

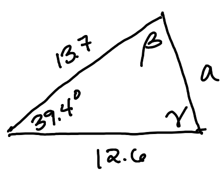
So, by substitution,

$A = \frac{1}{2}bc\sin\alpha$

SAS setup

Examples:

Find the area of the triangle with $\alpha = 39.4^\circ$, $b = 12.6$, and $c = 13.7$

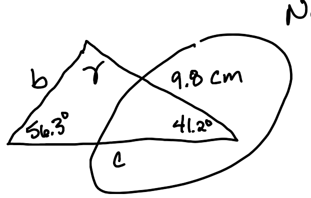


$A = \frac{1}{2}(12.6)(13.7)\sin 39.4^\circ$

$\approx \boxed{54.8 \text{ un}^2}$

Find the area of a triangle with $\alpha = 56.3^\circ$, $\beta = 41.2^\circ$, and $a = 9.8$ cm

Need SAS:



$\gamma = 180^\circ - (56.3^\circ + 41.2^\circ) = 82.5^\circ$

$\frac{\sin 56.3^\circ}{9.8} = \frac{\sin 82.5^\circ}{c}$

$c = \frac{9.8\sin 82.5^\circ}{\sin 56.3^\circ} \approx 11.7$

$A = \frac{1}{2}(9.8)(11.7)\sin 41.2^\circ$

$\approx \boxed{37.7 \text{ cm}^2}$

Heron's Formula

Using the law of cosines, it is possible to derive a formula for the area of a triangle that involves only the lengths of the sides of the triangle. The formula is known as "Heron's Formula" after Heron of Alexandria, who is believed to have discovered it around AD 75.

Heron's Formula: The area of a triangle with sides of lengths *a*, *b*, and *c* is given by:

$$A = \sqrt{S(S-a)(S-b)(S-c)}, \text{ where } S = (a+b+c)/2.$$

(*S* is called the semi-perimeter. It is half of the perimeter of the triangle.)

SSS setup

Examples:

Find the area of the triangle with $a = 12$ in, $b = 8$ in, and $c = 6$ in

$S = \frac{12+8+6}{2} = 13$

$A = \sqrt{13(13-12)(13-8)(13-6)} = \sqrt{455} \approx \boxed{21.3 \text{ in}^2}$

Find the area of a triangle with $a = 346$, $b = 234$, and $c = 422$

$S = \frac{346+234+422}{2} = 501$

$A = \sqrt{501(501-346)(501-234)(501-422)} = \boxed{40{,}471.9 \text{ un}^2}$