

## Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus is the theorem (idea) that connects the two branches of calculus—**differential calculus** (derivatives, differentials, rates of change, how fast we are going) and **integral calculus** (areas and accumulation, how far have we gone)

The Theorem comes in two parts:

### Part I:

If  $f$  is continuous on  $[a,b]$ , then the function  $F(x) = \int_a^x f(t) dt$  has a derivative at every point  $x$  in the interval  $[a,b]$ , namely  $f(x)$   $F'(x) = f(x)$  {  $x$  is the upper limit of integration } and

$$F'(x) = \frac{dF}{dx} = \frac{d}{dx} \int_a^x f(t) dt = f(x)$$

This theorem has unbelievable power! It tells us that the derivative of the integral is a function of the upper limit. The process of integration and differentiation are inverses!

How do we apply this?

Example: Find

$$\frac{d}{dx} \int_{\pi}^x \tan t dt \quad \text{answer is: } f = \tan x$$

$$\frac{d}{dx} \int_2^x (t^2 + t) dt \quad \text{answer is: } f = x^2 + x$$

What do you need to understand about the first part? The derivative of an integral is the integral function with the upper limit plugged in.

### Examples:

$$1. \frac{d}{dx} \int_0^x \sin t dt \qquad 2. \frac{d}{dx} \int_3^x \frac{1}{t+2} dt$$

What if the upper limit itself is a function?

We will need to use the chain rule.

$$F'(g) \cdot g' \text{ so } \frac{d}{dx} \int_1^{x^2} \sin t dt = \sin x^2 \cdot 2x = 2x \sin x^2$$

$$\text{In general } \frac{d}{dx} \int_c^u f(t) dt = f(u) \cdot \frac{du}{dx}$$

### Practice problems

Find  $\frac{dy}{dx}$

$$11. \quad y = \int_2^{5x} \frac{\sqrt{1+u^2}}{u} du$$

$$9. \quad y = \int_0^{x^2} e^{t^2} dt$$

What if the lower limit is  $x$ ?

### Look at Example 3

Now try:

$$13. \quad \frac{d}{dx} \int_x^6 \ln(1+t^2) dt$$

$$15. \quad y = \int_{x^3}^5 \frac{\cos t}{t^2 + 2} dt$$

What if both limits are functions of  $x$ ?

$$\begin{aligned} \frac{d}{dx} \int_x^{x^2} (3t^2 - 2t) dt &= \frac{d}{dx} \left( \int_0^{x^2} (3t^2 - 2t) dt - \int_0^x (3t^2 - 2t) dt \right) \\ &= \left( 3(x^2)^2 - 2(x^2) \right) 2x - (3x^2 - 2x) \\ &= (3x^4 - 2x^2)(2x) - 3x^2 + 2x \\ &= 6x^5 - 4x^3 - 3x^2 + 2x \end{aligned}$$

### Example 4

Chapter 7 will give us some idea of how to come up with the functions that produce this derivative. The correct function and its values can be easily computed with a calculator or computer.

Read and try: **Graphing the Function**  $\int_a^x f(t) dt$

### Exploration 1

### Exploration 2

### Fundamental Theorem of Calculus Part 2

The first part of the Fundamental Theorem of Calculus tells us how derivatives and integrals are related, namely, they are inverses of one another. The second part tells us how we can use this to evaluate integrals.

**If  $f$  is continuous at every point of  $[a,b]$ , and if  $F$  is any antiderivative of  $f$  on  $[a,b]$ , then**

$$\int_a^b f(x) dx = F(b) - F(a) \quad \text{where } F \text{ is any anti-derivative of } f$$

This part of the Fundamental Theorem is called the Integral Evaluation Theorem.

Let's take a look at the proof.

### Example:

$$\int_2^3 x^2 dx = \frac{x^3}{3} = \left[ \frac{3^3}{3} - \frac{2^3}{3} \right]_2 = \frac{19}{3}$$

Confirm with fnint

### Example:

$$\int_{-3}^6 (2x^3 - 4x + 1) dx = \frac{2x^4}{4} - \frac{4x^2}{2} + x = \frac{x^4}{2} - 2x^2 + x$$

### Example 5

Remember: the integral started out as a way to find the accumulated or net area under a curve. If function values are negative the integral will be negative. But area is always positive. When we say “area” under a curve we mean **total area**.

**Example 6:** Find the area under the curve of  $y = 4 - x^2$  from  $0 \leq x \leq 3$

**Example:** Find the area under the curve of  $y = x^3$  on  $[-1, 3]$

There are some general tips to help find the area.

#### **How to Find Total Area Analytically**

To find the area between the graph of  $y = f(x)$  and the x-axis over the interval  $[a, b]$  analytically,

1. Partition  $[a, b]$  with the zeros of  $f$  (Find the x-intercepts of the function – where function crosses x-axis)
2. Integrate  $f$  over each subinterval
3. Add the absolute values of the integrals

#### **How to Find Total Area Numerically**

To find the area between the graph of  $y = f(x)$  and the x-axis over the interval  $[a, b]$  numerically, evaluate  $\text{fnInt}(|f(x)|, x, a, b)$

### Example 7

### Example 8