Trigonometric Equations of the Quadratic Type

A General Strategy for Solving Trigonometric Equations

1. Know the solutions to $\sin x = 0$, $\cos x = 0$, and $\tan x = 0$.
2. Solve an equation involving multiple angles as if the equation had a single variable.
3. Simplify complicated equations by using identities. If possible, try to get an equation involving only one trigonometric function.
4. If possible, factor to get different trigonometric functions into separate factors.
5. For equations of the quadratic type, solve by factoring or by the quadratic formula.
6. Square each side of the equation, if necessary, so that identities involving squares can be applied. (Remember that this sometimes leads to extraneous solutions—check your answers.)

Examples: Find all real number solutions of the following equations.

a) $\sin x \tan x + \sin x = 0$

$\sin x(\tan x + 1) = 0$

$\sin x = 0$ or $\tan x + 1 = 0$

$\tan x = -1$

π 0

$\boxed{x = \pi k \text{ or } x = \frac{3\pi}{4} + \pi k}$

b) $\sin(2x) = \cos x$

$2\sin x \cos x = \cos x$

$2\sin x \cos x - \cos x = 0$

$\cos x(2\sin x - 1) = 0$

$\cos x = 0$ $\quad$ $2\sin x - 1 = 0$

$\sin x = \frac{1}{2}$

$\boxed{x = \frac{\pi}{2} + \pi k \text{ or } x = \frac{\pi}{6} + 2\pi k \text{ or } x = \frac{5\pi}{6} + 2\pi k}$

Examples: Find all angles in $[0°, 360°)$ that satisfy the following equations.

a) $6\cos^2\left(\frac{x}{2}\right) - 7\cos\left(\frac{x}{2}\right) + 2 = 0$

$u = \cos\left(\frac{x}{2}\right)$

$6u^2 - 7u + 2 = 0$

$6(2) = 12$
mult to 12
add to -7
-3 & -4

$6u^2 - 3u - 4u + 2 = 0$

$3u(2u-1) - 2(2u-1) = 0$

$(2u-1)(3u-2) = 0$

$u = \frac{1}{2}$ or $u = \frac{2}{3}$

$\cos\left(\frac{x}{2}\right) = \frac{1}{2}$ or $\cos\left(\frac{x}{2}\right) = \frac{2}{3}$

$\cos^{-1}\left(\frac{2}{3}\right) = 48.2°$

$2\left(\frac{x}{2}\right) = 2(60° + 360°k)$ $\quad$ $2\left(\frac{x}{2}\right) = 2(48.2° + 360°k)$

$2\left(\frac{x}{2}\right) = 2(300° + 360°k)$ $\quad$ $2\left(\frac{x}{2}\right) = 2(311.8° + 360°k)$

$48.2°$

$360° - 48.2° = 311.8°$

$x = 120° + 720°k$ $\quad$ $x = 96.4° + 720°k$

$x = 600° + 720°k$ $\quad$ $x = 623.6° + 720°k$

$\boxed{\{120°, 96.4°\}}$

b) $\cos\alpha - \sin^2\alpha = 0$

$\cos\alpha - (1 - \cos^2\alpha) = 0$

$\cos^2\alpha + \cos\alpha - 1 = 0$ $\quad$ $u = \cos\alpha$

$u^2 + u - 1 = 0$

doesn't factor, so
quadratic formula!

$u = \frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2(1)} = \frac{-1 \pm \sqrt{5}}{2}$

$\cos\alpha \approx .618$ or $\cancel{\cos\alpha \approx -1.618}$

impossible.
cosine is never less than -1

$\cos^{-1}(.618) \approx 51.8°$

$51.8°$

$360° - 51.8° = 308.2°$

$\boxed{\{51.8°, 308.2°\}}$

Examples: Find all solutions to the following equations in the interval $[0, 2\pi)$.

a) $(\sin\alpha - \cos\alpha)^2 = \left(\frac{1}{\sqrt{2}}\right)^2$

Square both sides
Caution! May introduce extraneous solutions

$\sin^2\alpha - 2\sin\alpha\cos\alpha + \cos^2\alpha = \frac{1}{2}$

$1 - 2\sin\alpha\cos\alpha = \frac{1}{2}$
$-1 \qquad\qquad -1$

$-\sin(2\alpha) = -\frac{1}{2}$

$\sin(2\alpha) = \frac{1}{2}$

$\frac{2\alpha}{2} = \frac{\frac{\pi}{6} + 2\pi k}{2}$ or $\frac{2\alpha}{2} = \frac{\frac{5\pi}{6} + 2\pi k}{2}$

$\alpha = \pi/12 + \pi k$ or $\alpha = 5\pi/12 + \pi k$

$\cancel{\pi/12}, 13\pi/12 \qquad 5\pi/12, \cancel{17\pi/12}$

Does not work in original equation

Does not work in original equation

$\boxed{\{13\pi/12, 5\pi/12\}}$

b) $\frac{\sin(2x)}{\cos(2x)} = \frac{3\cos(2x)}{\cos(2x)}$

$\tan^{-1}(3) \approx 1.25\text{ rad}$

$\tan(2x) = 3$

$\frac{2x}{2} = \frac{1.25 + \pi k}{2}$

$x \approx 0.62 + \frac{\pi}{2}k$

$\boxed{\{0.62, 2.20, 3.77, 5.34\}}$

Example: Find all angles in $[0°, 360°)$ that satisfy $\underbrace{\cos(2x)\cos(x) - \sin(2x)\sin x}_{\text{cosine sum identity}} = \sqrt{3}/2$.

$\cos(2x + x) = \sqrt{3}/2$

$\cos(3x) = \sqrt{3}/2$

$\frac{3x}{3} = \frac{30° + 360°k}{3}$ or $\frac{3x}{3} = \frac{330° + 360°k}{3}$

$x = 10° + 120°k$ or $x = 110° + 120°k$

$\boxed{\{10°, 130°, 250°, 110°, 230°, 350°\}}$

Example: Find all solutions in $[0, 2\pi)$ that satisfy $\underbrace{\sin x\cos(\pi/3) - \cos x\sin(\pi/3)}_{\sin\left(x - \frac{\pi}{3}\right)} = \sqrt{2}/2$.

$\sin(A - B) = \sin A\cos B - \cos A\sin B$

$\sin\left(x - \frac{\pi}{3}\right) = \frac{\sqrt{2}}{2}$

$x - \frac{\pi}{3} = \frac{\pi}{4} + 2\pi k$ or $x - \frac{\pi}{3} = \frac{3\pi}{4} + 2\pi k$
$+\frac{\pi}{3} \qquad +\frac{\pi}{3} \qquad\qquad +\frac{\pi}{3} \qquad +\frac{\pi}{3}$

$x = \frac{7\pi}{12} + 2\pi k$ or $x = \frac{13\pi}{12} + 2\pi k$

$\boxed{\left\{\frac{7\pi}{12}, \frac{13\pi}{12}\right\}}$