Definite Integrals

Estimating areas using finite sums is one way of calculating accumulations. Earlier we said differential calculus deals with rates of change. Integral calculus deals with accumulations. The **definite integral** is a way of calculating the area under a curve.

We estimated areas using a finite number of rectangles or volumes that we added together. What we did in section 6.1 was rather tedious work. We can use “sigma” notation to write large sums in a compact form.

$$\sum_{k=1}^{n} a_n = a_1 + a_2 + a_3 + ... + a_n$$

Greek letter sigma means sum. K is the index or what term we are starting with/on.

One approximation (in 6.1) that we used was

$\frac{1}{2}\left(\frac{1}{4}\right)+\frac{1}{2}(1)+\frac{1}{2}\left(\frac{9}{4}\right)+\frac{1}{2}(4)+\frac{1}{2}\left(\frac{25}{4}\right)+\frac{1}{2}(9)=\sum_{1}^{6}\frac{1}{2}(k)^2$ RRAM method $y = x^2$ with $\Delta x = ½$

The sums that we are interested in are called “Riemann” sums, named after Georg Riemann who developed this method for finding the area under a curve. Riemann’s idea was to break an interval into arbitrary rectangles that when added together, approximate the area under the curve.

First: we need a function defined on an interval.
Second: we need to “partition” the interval. A partition breaks an interval into subintervals
Third: select a number in each subinterval and compute the functions value at that point $f(c_k)$
Fourth: Make a rectangle in each subinterval that has width Δx_n and height $f(c_k)$
Fifth: Sum the area of all the rectangles

Since there are n subintervals, the sum of those n rectangles is

$$S_n = \sum_{k=1}^{n} f\left(c_k\right) \cdot \Delta x_k$$

$$(height) \cdot (width)$$

$$(function) \cdot (dx)$$

The sum, which depends on the partition and on the c_k is called a Riemann Sum for f on [a,b]
Riemann sums are approximations! How could we make this better?
What would happen if the partitions became finer and finer? The rectangles would become smaller and smaller. What would happen to the Riemann Sum?

Look at 6.15

If we think of Riemann sums as LRAM, MRAM, and RRAM all of these converged to a common limit as we refined the partition.
This is true of Riemann Sums. All Riemann sums converge to a common value as long as each Δx_k tends to zero. We can guarantee that the subintervals will go to zero by saying that the “norm” of the partition will tend to zero. This is noted $\|P\| \to 0$ “the magnitude of the longest subinterval will tend to zero.”

Definition of Definite Integral as a Limit of Riemann Sums
Let f be a function defined on a closed interval [a,b]. For any partition P of [a,b], let numbers c_k be chosen arbitrarily in the subintervals $[x_{k-1}, x_k]$. If there exists a number I such that

$\lim_{\|P \to 0\|} \sum_{k=1}^{n} f(c_k)\Delta x_k = I$ no matter how P and the c_k's are chosen, then f is **integrable** on [a,b] and I is the **definite integral** of f over [a,b]

In other words the definite integral is the area under a curve on a closed interval

If a function has an **integral** the function is said to be **integrable**.

All continuous functions are integrable.

Theorem 1: The Existence of Definite Integrals
All continuous functions are integrable. That is, if a function f is continuous on an interval [a, b], then its definite integral over [a, b] exists.

Because continuous functions are so nice, we can add some uniformity.

1st: Make n subintervals each one $\Delta x = \frac{b-a}{n}$

2nd Choose c_k in each interval
3rd

$$I = \lim_{n \to \infty} \sum_{k=1}^{n} f(c_k) \cdot \Delta x$$

$(height) \cdot (width)$ dx's are uniform

$(function\,value) \cdot (dx)$

We now change notation for the last time

$$I = \lim_{n \to \infty} \sum_{k=1}^{n} f(c_k) \cdot \Delta x = {}_a S^b f(x)dx = \int_a^b f(x)dx$$

Sum of each rectangle (f(x))(dx) from a to b = "integral from a to b of f(x)dx

Look at integral notation on page 281. Label each part.

$$\int_a^b f(x)dx$$

No matter how we represent the integral, it is the same number, defined as a limit of Riemann sums. Since it does not matter what letter we use to run from a to b the variable of integration is called a dummy variable.

Example 1 Using the Notation

Definition: Area under a Curve (as a Definite Integral)
If y=f(x) is nonnegative and integrable over a closed interval [a,b], then the **area under the curve y=f(x) from a to b** is the integral of f from a to b.

$$A = \int_a^b f(x)dx$$

This definition works both ways: We can use integrals to calculate areas and we can use areas to calculate integrals.

Some areas are difficult
Some areas are easy.

#16 $\int_{-4}^{0} \sqrt{16 - x^2}dx$

#19 $\int_{-1}^{1} \left(2 - |x|\right) dx$

Can area ever be negative?

What if we have a function that has negative function values?

Example: $\int_{-2}^{2} -\sqrt{4 - x^2}\, dx$

$$Area = -\int_a^b f(x)dx \quad when \quad f(x) \le 0$$

If an integrable function y=f(x) has both positive and negative values on an interval [a,b], then the Riemann sums for f on [a,b] add areas of rectangles that lie above the x-axis to the negatives of areas of rectangles that lie below the x-axis. The value of the integral is the area above the x-axis minus the area below.

$\int_a^b f(x)dx$ = (area above the x-axis) – (area below the x-axis)

Exploration 1

One of the easiest types of functions to integrate is a **constant function.**

Theorem 2 The Integral of a Constant
If f(x) = c, where c is a constant, on the interval [a,b], then

$$\int_a^b f(x)dx = \int_a^b c\,dx = c(b-a)$$

Examples:

#7 $\int_{-2}^{1} 5\,dx$

#8 $\int_{3}^{7} (-20)\,dx$

In each case we took the constant and multiplied it by the length of the interval. c(b-a)

Example 3

Just like your calculator will do numeric derivatives, it will also do numeric integrals. It calculates Riemann sums very quickly.

Integrals on a Calculator
Using a ti-84….find "fnInt" Math key #9. Paste it to the home screen (f(x), x, a, b)

Example 4

Now try #34 $3+2\int_0^{\frac{\pi}{3}} \tan x\ dx$

Discontinuous Integrable Functions
Can functions that are discontinuous be integrable? Some can.

Example: Graph

$$f(x) = \frac{|x|}{x}$$

$$Find\ \int_{-1}^{2} \frac{|x|}{x} dx$$

Why does this work?
Exploration 2