Precalculus – 6.2 Notes

Solving Trigonometric Equations

Basic steps for solving cos *x* = *a*:

1. **Find all the angles on the unit circle (on $[0, 2\pi]$) that satisfy the equation.** If a is not a unit circle value, use $\cos^{-1}(a)$ to find one of the angles, then figure out any other angles with the same cosine (draw the angle and figure out the other angle with the same *x*-coordinate on the unit circle).
2. **Add or subtract $2\pi k$ from each angle, where k is any integer.**

Basic steps for solving sin *x* = *a*:

1. **Find all the angles on the unit circle (on $[0, 2\pi]$) that satisfy the equation.** If a is not a unit circle value, use $\sin^{-1}(a)$ to find one of the angles (if you get a negative angle, add 2π), then figure out any other angles with the same sine (draw the angle and figure out the other angle with the same *y*-coordinate on the unit circle).
2. **Add or subtract $2\pi k$ from each angle, where k is any integer.**

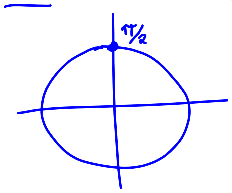
Basic steps for solving tan *x* = *a*:

1. **Find one angle on the unit circle that satisfies the equation.** If a is not a unit circle value, use $\tan^{-1}(a)$ to find an angle. You don't need to find another angle, because unlike sine and cosine, tangent repeats at regular intervals.
2. **Add or subtract multiples of π from each angle.** (Tangent repeats every π instead of every 2π like sine and cosine).

Examples: Find all real numbers (that means radians) that satisfy each equation.

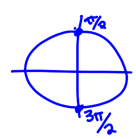
a) $\sin x = 1$

$\boxed{x = \frac{\pi}{2} + 2\pi k}$



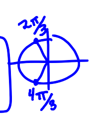
b) $\cos x = 0$

$\boxed{x = \frac{\pi}{2} + \pi k}$



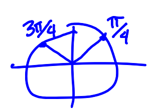
c) $\cos x = -1/2$

$x = \frac{2\pi}{3} + 2\pi k$ or $x = \frac{4\pi}{3} + 2\pi k$



d) $\sin x = \sqrt{2}/2$

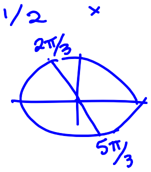
$x = \frac{\pi}{4} + 2\pi k$ or $x = \frac{3\pi}{4} + 2\pi k$



e) $\tan x = -\sqrt{3}$

$-\frac{\sqrt{3}/2}{1/2}$ $\quad \frac{y}{x}$

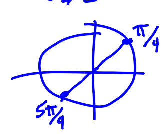
$\boxed{x = \frac{2\pi}{3} + \pi k}$



f) $\tan x = 1$

$\frac{\sqrt{2}/2}{\sqrt{2}/2}$

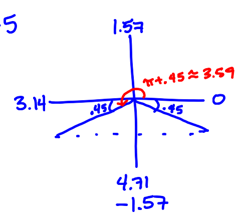
$\boxed{x = \frac{\pi}{4} + \pi k}$



g) $\sin x = -.4375$

$\sin^{-1}(-.4375) \approx -.45$

$x \approx -.45 + 2\pi k$ or $x \approx 3.59 + 2\pi k$

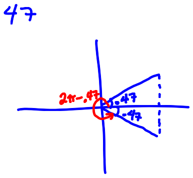


h) $\cos x = .8913$

$\cos^{-1}(.8913) \approx .47$

$2\pi - .47 \approx 5.81$

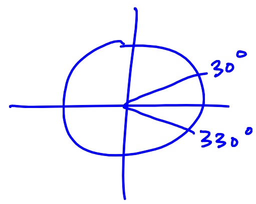
$x \approx .47 + 2\pi k$ or $x \approx 5.81 + 2\pi k$



Examples: Find all angles in $[0°,\ 360°]$ that satisfy each equation

a) $\cos x = \sqrt{3}/2$

$\boxed{\{30°, 330°\}}$



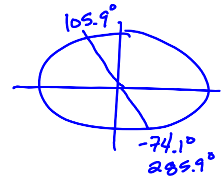
b) $\tan x = -3.5$

$\tan^{-1}(-3.5) \approx -74.1°$

$-74.1° + 180° = 105.9°$

$105.9° + 180° = 285.9°$

$\boxed{\{105.9°, 285.9°\}}$



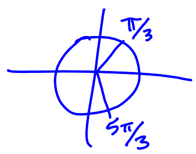
Sometimes, you have to do a bit of algebra before you can use the techniques above.

a) Solve $2\cos\alpha - 1 = 0$ for $0 \le \alpha \le 2\pi$.

$2\cos\alpha = 1$

$\cos\alpha = \frac{1}{2}$

$\boxed{\{\frac{\pi}{3}, \frac{5\pi}{3}\}}$



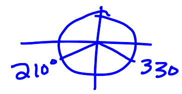
b) Solve $3\sin(\beta) + 6 = 5\sin(\beta) + 7$ for $0° \le \beta \le 360°$.

$-3\sin\beta - 7 \quad -3\sin\beta - 7$

$-1 = 2\sin\beta$

$\sin\beta = -\frac{1}{2}$

$\boxed{\{210°, 330°\}}$



Solving Multiple-Angle Equations

Often, equations involve expressions like $\sin 2x,\ \cos 3\alpha,$ or $\tan(x/2)$, all of which involve multiples of the variable rather than a single variable.

1. Solve for the multiple variable just as we would solve for a single variable. (eg. Solve for $2x$)
2. Multiply or divide to get the single variable in the last step. (eg. Divide by 2 to solve for x)

Example: Find all solutions in degrees to $\sin 2\alpha = \sqrt{3}/2$.

Solve for 2α: $\frac{2\alpha}{2} = \frac{60° + 360°k}{2}$ or $\frac{2\alpha}{2} = \frac{120° + 360°k}{2}$

Divide by 2 to get α: $\boxed{\alpha = 30° + 180°k \quad \text{or} \quad \alpha = 60° + 180°k}$

Example: Find all solutions to $\tan(4x) = 1$ in the interval $(0, \pi)$.

Solve for $4x$: $\frac{4x}{4} = \frac{\frac{\pi}{4} + \pi k}{4}$

Divide by 4 to get x: $x = \frac{\pi}{16} + \frac{\pi}{4}k$

List: $\boxed{\{\frac{\pi}{16}, \frac{5\pi}{16}, \frac{9\pi}{16}, \frac{13\pi}{16}\}}$

Example: Find all real number solutions to $\cos(x/2) = -1/2$.

Solve for $\frac{x}{2}$: $2\left(\frac{x}{2}\right) = 2\left(\frac{2\pi}{3} + 2\pi k\right)$ or $2\left(\frac{x}{2}\right) = 2\left(\frac{4\pi}{3} + 2\pi k\right)$

Multiply by 2 to get x: $\boxed{x = \frac{4\pi}{3} + 4\pi k \quad \text{or} \quad x = \frac{8\pi}{3} + 4\pi k}$

Example: Find all solutions to $\sec(3x) = 2\sqrt{3}/3$ in the interval $(0°, 360°)$.

$\cos(3x) = \sqrt{3}/2$

Solve for $3x$: $\frac{3x}{3} = \frac{30° + 360°k}{3}$ or $\frac{3x}{3} = \frac{330° + 360°k}{3}$

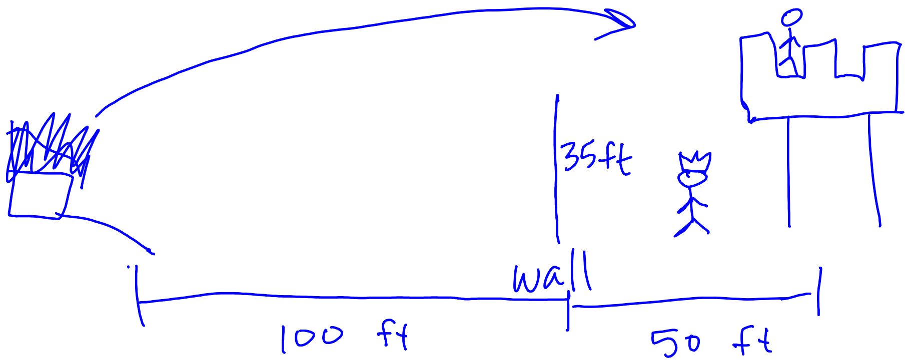
Divide by 3 to get x: $x = 10° + 120°k$ or $x = 110° + 120°k$

List: $\boxed{\{10°, 130°, 250°, 110°, 230°, 350°\}}$

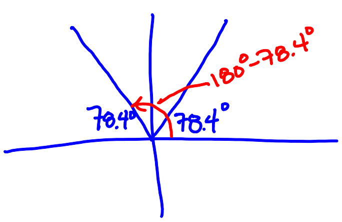
The Path of a Projectile

The distance d (in feet) traveled by a projectile fired from the ground with an angle of elevation θ is related to the initial velocity v_0 (in ft/sec) by the equation $v_0^2 \sin 2\theta = 32d$. If the projectile is fired from the origin into the first quadrant, then the x- and y-coordinates (in feet) of the projectile at time t (in seconds) are given by $x = v_0 t \cos\theta$ and $y = -16t^2 + v_0 t \sin\theta$.

Example: A catapult is placed 100 feet from the castle wall, which is 35 feet high. A soldier wants a burning bale of hay to clear the top of the wall and land 50 feet inside the castle wall. If the initial velocity of the bale is 70 ft/sec, then at what angle should the bale of hay be launched so that it will travel 150 feet and pass over the castle wall?



$$V_0^2 \sin 2\theta = 32d \qquad V_0 = 70 \text{ ft/sec}$$
$$70^2 \sin 2\theta = 32(150) \qquad d = 150 \text{ ft}$$
$$\sin 2\theta = \frac{32(150)}{70^2}$$
$$\sin 2\theta = \frac{4800}{4900}$$



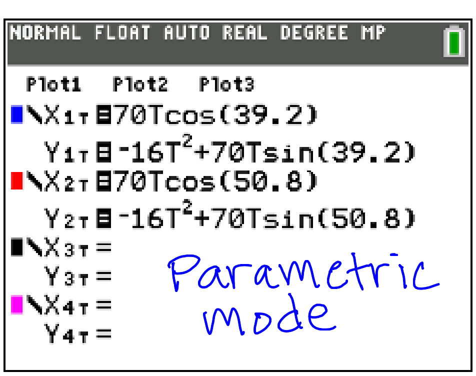
$$\sin^{-1}\left(\frac{48}{49}\right) \approx 78.4^\circ$$
$$180^\circ - 78.4^\circ \approx 101.6^\circ$$

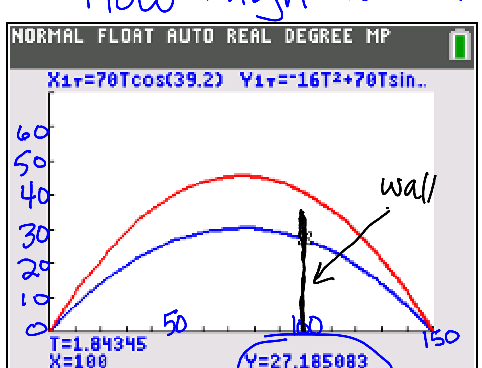
$$\frac{2\theta}{2} = \frac{78.4^\circ}{2} \quad \text{or} \quad \frac{2\theta}{2} = \frac{101.6^\circ}{2}$$
$$\theta \approx 39.2^\circ \quad \text{or} \quad \theta \approx 50.8^\circ$$

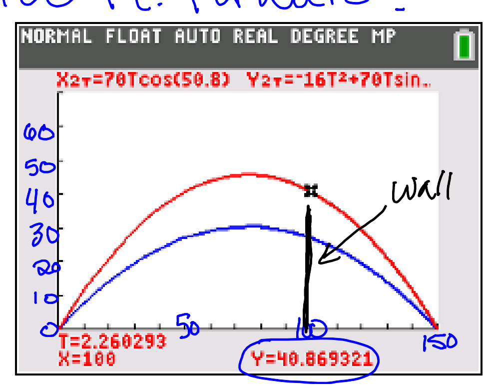
Will both clear the wall?

Use x & y equations to find out.

How high when 100 ft. forward?







Too low @ 39.2°

Clears wall @ 50.8°