Double-Angle and Half-Angle Identities

Using the sum identities from the last section, we can derive more formulas called the double-angle identities:

$\sin(2x) = \sin(x+x) =$

$\cos(2x) = \cos(x+x) =$

We can use Pythagorean Identities to derive two more identities for $\cos(2x)$.

$\cos(2x) =$ $\cos(2x) =$

$\tan(2x) = \tan(x+x) =$

Example: Use the double angle identities to verify that $\cos(3x) = \cos^3 x - 3\cos x \sin^2 x$ is an identity.

We can use the double-angle identities to derive identities for $\sin(x/2)$, $\cos(x/2)$, and $\tan(x/2)$. We call these the half-angle identities.

To get identities for $\cos(x/2)$ and $\sin(x/2)$, we solve two of the identities for $\cos(2x)$ for $\sin x$ and $\cos x$.

$$2\cos^2 x - 1 = \cos(2x)$$
$$\cos^2 x = \frac{1+\cos(2x)}{2}$$
$$\cos x = \pm\sqrt{\frac{1+\cos(2x)}{2}}$$

$$1 - 2\sin^2 x = \cos(2x)$$
$$-2\sin^2 x = \cos(2x) - 1$$
$$\sin^2 x = \frac{1-\cos(2x)}{2}$$
$$\sin x = \pm\sqrt{\frac{1-\cos(2x)}{2}}$$

Since these equations work for any angle, they also work if we replace x by $(x/2)$:

$$\cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}} \qquad \sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$$

We can then use these formulas to derive formulas for $\tan\frac{x}{2}$:

$$\tan\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}} \qquad \tan\frac{x}{2} = \frac{\sin x}{1+\cos x} \qquad \tan\frac{x}{2} = \frac{1-\cos x}{\sin x}$$

Examples: Use the half-angle identities to find the exact values of the following:

$\sin(67.5°)$ $\qquad$ $\cos(11\pi/12)$ $\qquad$ $\tan(-\pi/8)$

Example: Prove that $\sin^2\left(\frac{x}{2}\right)\cos^2\left(\frac{x}{2}\right) = \frac{\sin^2 x}{4}$ is an identity.

Example: Find $\sin\alpha$, $\cos\alpha$, and $\tan\alpha$ if $\sin(2\alpha) = -1/3$ and $\pi < 2\alpha < 3\pi/2$.

Example: Find $\sin\alpha$, $\cos\alpha$, and $\tan\alpha$ if $\sin(\alpha/2) = 12/13$ and $\pi/2 < \alpha/2 < \pi$.