Sum and Difference Identities for Cosine

Often, an angle can be expressed as a sum or difference of two angles for which we know the exact values of the trigonometric functions. We can use sum and difference identities to find the exact values of the trigonometric functions of our angle of interest.

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$
$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$

Mnemonic: "Cosine changes the silly sign."

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$
$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$

Mnemonic: "Sine can't change signs."

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}\qquad \tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

Examples: Write the following angles as sums or differences of two other angles whose trigonometric functions can be calculated exactly:

$105^\circ = 60^\circ+45^\circ$, $135^\circ-30^\circ$

$15^\circ = 45^\circ-30^\circ$, $60^\circ-45^\circ$

$345^\circ = 300^\circ+45^\circ$

$195^\circ = 60^\circ+135^\circ$, $225^\circ-30^\circ$

$\frac{7\pi}{12} = \frac{3\pi}{12}+\frac{4\pi}{12} = \frac{\pi}{4}+\frac{\pi}{3}$

$\frac{\pi}{12} = \frac{4\pi}{12}-\frac{3\pi}{12} = \frac{\pi}{3}-\frac{\pi}{4}$; $\frac{3\pi}{12}-\frac{2\pi}{12} = \frac{\pi}{4}-\frac{\pi}{6}$

$-\frac{\pi}{12} = \frac{\pi}{4}-\frac{\pi}{3}$; $\frac{\pi}{6}-\frac{\pi}{4}$

$\frac{5\pi}{12} = \frac{2\pi}{12}+\frac{3\pi}{12} = \frac{\pi}{6}+\frac{\pi}{4}$

Find the exact value of $\cos(15^\circ)$.

$$\cos(60^\circ-45^\circ) = \cos 60^\circ\cos 45^\circ + \sin 60^\circ \sin 45^\circ = \left(\frac{1}{2}\right)\left(\frac{\sqrt2}{2}\right)+\left(\frac{\sqrt3}{2}\right)\left(\frac{\sqrt2}{2}\right) = \frac{\sqrt2}{4}+\frac{\sqrt6}{4} = \frac{\sqrt2+\sqrt6}{4}$$

Find the exact value of $\cos(11\pi/12)$.

$9\pi/12 + 2\pi/12$

$$\cos\left(\frac{3\pi}{4}+\frac{\pi}{6}\right) = \cos\left(\frac{3\pi}{4}\right)\cos\left(\frac{\pi}{6}\right)-\sin\left(\frac{3\pi}{4}\right)\sin\left(\frac{\pi}{6}\right) = \left(-\frac{\sqrt2}{2}\right)\left(\frac{\sqrt3}{2}\right)-\left(\frac{\sqrt2}{2}\right)\left(\frac{1}{2}\right)$$
$$\frac{-\sqrt6-\sqrt2}{4}$$

Find the exact value of $\sin(165^\circ)$.

$$\sin(120^\circ+45^\circ) = \sin(120^\circ)\cos(45^\circ)+\cos(120^\circ)\sin(45^\circ) = \left(\frac{\sqrt3}{2}\right)\left(\frac{\sqrt2}{2}\right)+\left(-\frac{1}{2}\right)\left(\frac{\sqrt2}{2}\right) = \frac{\sqrt6-\sqrt2}{4}$$

Find the exact value of $\sin(-5\pi/12)$.

$-\frac{3\pi}{12} - \frac{2\pi}{12}$

$$\sin\left(-\frac{\pi}{4}-\frac{\pi}{6}\right) = \sin\left(-\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right)-\cos\left(-\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right) = \left(-\frac{\sqrt2}{2}\right)\left(\frac{\sqrt3}{2}\right)-\left(\frac{\sqrt2}{2}\right)\left(\frac{1}{2}\right)$$
$$\frac{-\sqrt6-\sqrt2}{4}$$

Find the exact value of $\tan(15^\circ)$.

$$\tan(60^\circ-45^\circ) = \frac{\tan 60^\circ-\tan 45^\circ}{1+\tan 60^\circ\tan 45^\circ} = \frac{\sqrt3-1}{1+(\sqrt3)(1)} = \boxed{\frac{\sqrt3-1}{1+\sqrt3}}$$

Find the exact value of $\tan(5\pi/12)$.

$\frac{2\pi}{12}+\frac{3\pi}{12}$

$$\tan\left(\frac{\pi}{6}+\frac{\pi}{4}\right) = \frac{\tan\frac{\pi}{6}+\tan\frac{\pi}{4}}{1-\tan\frac{\pi}{6}\tan\frac{\pi}{4}} = \left(\frac{\frac{1}{\sqrt3}+1}{1-\frac{1}{\sqrt3}\cdot 1}\right)\left(\frac{\sqrt3}{\sqrt3}\right) = \boxed{\frac{1+\sqrt3}{\sqrt3-1}}$$

Use appropriate identities to simplify each expression:

$\sin 49^\circ\cos 4^\circ+\cos 49^\circ\sin 4^\circ$

$$\sin(49^\circ+4^\circ) = \boxed{\sin(53^\circ)}$$

$\cos(x)\cos(-5x)+\sin(-x)\sin(5x)$

odd/even: $\cos(x)\cos(5x)-\sin(x)\sin(5x)$

$\cos(x+5x)$

$\boxed{\cos(6x)}$

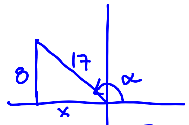
Find the exact value of $\sin(\alpha+\beta)$ if $\sin\alpha = \overset{y/r}{8/17}$ and $\cos\beta = \overset{x/r}{-1/3}$. α is in Quadrant II and β is in Quadrant III.

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$= \left(\frac{8}{17}\right)\left(-\frac{1}{3}\right) + \left(-\frac{15}{17}\right)\left(-\frac{2\sqrt{2}}{3}\right)$

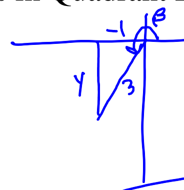
$= -\frac{8}{51} + \frac{30\sqrt{2}}{51}$

$= \boxed{\frac{-8+30\sqrt{2}}{51}}$



$x = -\sqrt{17^2-8^2} = -15$

$\cos\alpha = -\frac{15}{17}$



$y = -\sqrt{3^2-1^2}$

$= -\sqrt{8} = -2\sqrt{2}$

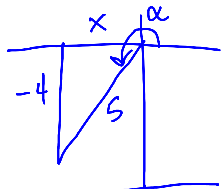
$\sin\beta = -\frac{2\sqrt{2}}{3}$

Find the exact value of $\cos(\alpha-\beta)$ if $\sin\alpha = \overset{y/r}{-4/5}$ and $\cos\beta = \overset{x/r}{12/13}$. α is in Quadrant III. β is in Quadrant IV.

$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

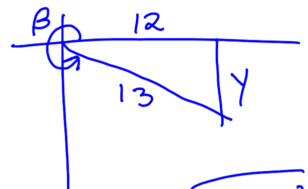
$= \left(-\frac{3}{5}\right)\left(\frac{12}{13}\right) + \left(-\frac{4}{5}\right)\left(-\frac{5}{13}\right)$

$= \frac{-36}{65} + \frac{20}{65} = \boxed{\frac{-16}{65}}$



$x = -\sqrt{5^2-4^2} = -3$

$\cos\alpha = -\frac{3}{5}$



$y = -\sqrt{13^2-12^2} = -5$

$\sin\beta = -\frac{5}{13}$

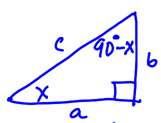
Cofunction Identities

Sine and cosine are cofunctions, secant and cosecant are cofunctions, and tangent and cotangent are cofunctions. Each function can be related to its cofunction by a simple cofunction identity.

$\sin\left(\frac{\pi}{2}-x\right) = \cos x \qquad \cos\left(\frac{\pi}{2}-x\right) = \sin x$

$\tan\left(\frac{\pi}{2}-x\right) = \cot x \qquad \cot\left(\frac{\pi}{2}-x\right) = \tan x$

$\sec\left(\frac{\pi}{2}-x\right) = \csc x \qquad \csc\left(\frac{\pi}{2}-x\right) = \sec x$



$\sin x = \frac{b}{c}$

$\cos(90°-x) = \frac{b}{c}$, so $\sin x = \cos(90°-x)$

$\sin(90°-x) = \cos x \qquad \cos(90°-x) = \sin x$

$\tan(90°-x) = \cot x \qquad \cot(90°-x) = \tan x$

$\sec(90°-x) = \csc x \qquad \csc(90°-x) = \sec x$

eg.) $\sin 20° = \cos 70°$

$\tan 62° = \cot 28°$

$\sec 7° = \csc 83°$

Examples: Use appropriate identities to simplify each expression.

$\cos(10°)\cos(20°) + \cos(-80°)\sin(-20°)$

odd/even: $\cos(10°)\cos(20°) + \cos(80°)(-\sin 20°)$

$\cos(10°)\cos(20°) - \cos(80°)\sin(20°)$

cofunction: $\cos(80°) = \sin(10°)$

$\cos(10°)\cos(20°) - \sin(10°)\sin(20°)$

$\cos(10°+20°) = \cos(30°) = \boxed{\frac{\sqrt{3}}{2}}$

$\sin(65°)\sin(5°) + \sin(-25°)\sin(-85°)$

odd/even: $\sin(65°)\sin(5°) + (-\sin 25°)(-\sin 85°)$

$\sin(65°)\sin(5°) + \sin(25°)\sin(85°)$

cofunction: $\sin 5° = \cos 85° \qquad \sin 25° = \cos 65°$

$\sin(65°)\cos(85°) + \cos(65°)\sin(85°)$

$\sin(65°+85°) = \sin(150°)$

$= \boxed{\frac{1}{2}}$

Examples: Prove that the following is an identity.

$$\frac{\cos(x+y)}{\cos(x-y)} = \frac{\cot y - \tan x}{\cot y + \tan x}$$

$$\frac{\cos x\cos y - \sin x\sin y}{\cos x\cos y + \sin x\sin y}$$

$$\frac{\frac{\cos y}{\sin y} - \frac{\sin x}{\cos x}}{\frac{\cos y}{\sin y} + \frac{\sin x}{\cos x}} \cdot \frac{\cos x\sin y}{\cos x\sin y}$$

$$\frac{\cos x\cos y - \sin x\sin y}{\cos x\cos y + \sin x\sin y}$$

Q.E.D.