Basic Trigonometric Identities

An ***equation*** is any mathematical statement involving an equal sign. There are three types of equations:

- ***Contradictions*** are equations that are never true, like $0=1$, or $x+5=x-7$.
- ***Conditional equations*** are equations that are sometimes true - true only for certain values of the variable(s) - like $x+5=7$, or $\sin\theta=\sqrt{3}/2$.
- ***Identities*** are equations that are true for all possible values of the variables, like $x+y=y+x$, or $A^2-B^2=(A+B)(A-B)$ or $\csc\theta=1/\sin\theta$.

Many trigonometric identities can be derived quickly from the *x*, *y*, *r* definitions of the trigonometric functions.

Reciprocal Identities:

$$\sin=\frac{1}{\csc}\qquad \cos=\frac{1}{\sec}\qquad \tan=\frac{1}{\cot}$$

$$\csc=\frac{1}{\sin}\qquad \sec=\frac{1}{\cos}\qquad \cot=\frac{1}{\tan}$$

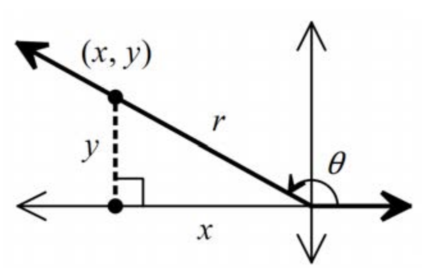
Tangent and Cotangent Quotient Identities:

$$\tan=\frac{\sin}{\cos}\qquad \cot=\frac{\cos}{\sin}$$

$$\frac{\frac{y}{r}}{\frac{x}{r}}=\frac{y}{\cancel{r}}\cdot\frac{\cancel{r}}{x}=\frac{y}{x}$$

The Fundamental Identity

Remember that by definition, $\sin\theta=y/r$, $\cos\theta=x/r$, and by the Pythagorean Theorem, $x^2+y^2=r^2$.



$$\sin^2\theta+\cos^2\theta=\left(\frac{y}{r}\right)^2+\left(\frac{x}{r}\right)^2$$

$$=\frac{y^2}{r^2}+\frac{x^2}{r^2}$$

$$=\frac{y^2+x^2}{r^2}$$

$$=\frac{r^2}{r^2}$$

$$=\underline{1}$$

sin²θ means (sin θ)²

$y^2+x^2=r^2$ by Pythagorean Theorem

☆ **The Fundamental (Pythagorean) Identity:** $\sin^2\theta+\cos^2\theta=\underline{1}$

Pythagorean Identities: We can use the fundamental identity to derive two more identities. Together, these three identities are called Pythagorean Identities because they are derived from the Pythagorean Theorem:

$$\sin^2\theta+\cos^2\theta=1$$

$$\frac{\sin^2\theta+\cos^2\theta}{\sin^2\theta}=\frac{1}{\sin^2\theta}$$

$$\frac{\sin^2\theta}{\sin^2\theta}+\frac{\cos^2\theta}{\sin^2\theta}=\frac{1}{\sin^2\theta}$$

☆ $\underline{1}+\underline{\cot^2\theta}=\underline{\csc^2\theta}$

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$$\frac{\sin^2\theta}{\cos^2\theta}+\frac{\cos^2\theta}{\cos^2\theta}=\frac{1}{\cos^2\theta}$$

☆ $\underline{\tan^2\theta}+\underline{1}=\underline{\sec^2\theta}$

Simplifying Expressions

We can use the identities above to simplify trigonometric expressions. One of the most common strategies is to start by rewriting the expression in terms of sines and/or cosines, then simplify from there.

Examples: Simplify the following.

a) $\dfrac{\tan x}{\sec x} = \dfrac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x}}$

$= \dfrac{\sin x}{\cancel{\cos x}} \cdot \dfrac{\cancel{\cos x}}{1}$

$= \boxed{\sin x}$

b) $\sin\alpha + \cot\alpha\cos\alpha$

$= \sin\alpha + \left(\dfrac{\cos\alpha}{\sin\alpha}\right)\cos\alpha$

$= \sin\alpha + \dfrac{\cos^2\alpha}{\sin\alpha}$

$= \sin\alpha\left(\dfrac{\sin\alpha}{\sin\alpha}\right) + \dfrac{\cos^2\alpha}{\sin\alpha}$

$= \dfrac{\sin^2\alpha + \cos^2\alpha}{\sin\alpha}$

$= \dfrac{1}{\sin\alpha} = \csc\alpha$

c) $\dfrac{\tan\theta\csc\theta}{\sec\theta}$

$\dfrac{\frac{\cancel{\sin\theta}}{\cos\theta} \cdot \frac{1}{\cancel{\sin\theta}}}{\frac{1}{\cos\theta}} = \dfrac{\frac{1}{\cos\theta}}{\frac{1}{\cos\theta}}$

$= \dfrac{1}{\cancel{\cos\theta}} \cdot \dfrac{\cancel{\cos\theta}}{1} = \boxed{1}$

Using Identities to Find Function Values

We know how to draw a triangle to find missing function values, but we can also find missing function values using identities.

Example: If $\tan\alpha = -2/3$ and α is in quadrant IV, use identities to find the values of the remaining five trigonometric functions.

$\cos\alpha$ & $\sec\alpha$ are positive
$\sin\alpha$, $\csc\alpha$, $\tan\alpha$, $\cot\alpha$ are negative

$\cot\alpha = \dfrac{1}{\tan\alpha} = \dfrac{1}{-2/3} = \boxed{-\dfrac{3}{2}}$

$\tan^2\alpha + 1 = \sec^2\alpha$

$\left(-\frac{2}{3}\right)^2 + 1 = \sec^2\alpha$

$\frac{4}{9} + 1 = \sec^2\alpha$

$\sqrt{\frac{13}{9}} = \sqrt{\sec^2\alpha}$

$\boxed{\sec\alpha = \frac{\sqrt{13}}{3}}$

$\cos\alpha = \dfrac{1}{\sec\alpha} = \dfrac{1}{\sqrt{13}/3} = \boxed{\dfrac{3}{\sqrt{13}}}$

$\tan\alpha = \dfrac{\sin\alpha}{\cos\alpha}$

$\left(-\frac{2}{3}\right) = \left(\dfrac{\sin\alpha}{3/\sqrt{13}}\right)$

$\sin\alpha = \left(-\frac{2}{3}\right)\left(\frac{3}{\sqrt{13}}\right)$

$\boxed{\sin\alpha = -\frac{2}{\sqrt{13}}}$

$\csc\alpha = \dfrac{1}{\sin\alpha} = \dfrac{1}{-2/\sqrt{13}} = \boxed{-\sqrt{13}/2}$

Multiplying and Factoring Polynomials Involving Trigonometric Functions

We must often multiply or factor expressions involving trigonometric functions when we simplify or verify identities or solve trigonometric equations.

Examples:

a) Multiply $(1+\tan x)(1-\tan x)$

$1 - \cancel{\tan x} + \cancel{\tan x} - \tan^2 x$

$= \boxed{1 - \tan^2 x}$

b) Multiply $(2\sin x + 1)^2$

$(2\sin x + 1)(2\sin x + 1)$

$4\sin^2 x + 2\sin x + 2\sin x + 1$

$\boxed{4\sin^2 x + 4\sin x + 1}$

c) Factor $2\sin x\cos x + \cos x$

$\boxed{\cos x(2\sin x + 1)}$ GCF

d) Factor $\sec^2 x - \tan^2 x$

Diff of squares
$A^2 - B^2 = (A+B)(A-B)$

$(\sec x + \tan x)(\sec x - \tan x)$

e) Factor $\sin^2 x + \sin x - 2$

$u = \sin x$ $\quad u^2 + u - 2$

$(u+2)(u-1)$

$\boxed{(\sin x + 2)(\sin x - 1)}$

f) Factor $3\cos^2 x - 7\cos x - 6$

$3(-6) = -18$ mult to
add to -7
2 & -9

$u = \cos x$ $\quad 3u^2 - 7u - 6$

$3u^2 + 2u - 9u - 6$

$u(3u+2) - 3(3u+2)$

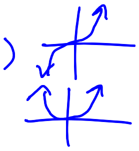
$(3u+2)(u-3)$

$\boxed{(3\cos x + 2)(\cos x - 3)}$

Odd and Even Identities

Odd functions have graphs that are symmetric with respect to the origin. $(y=x^3)$

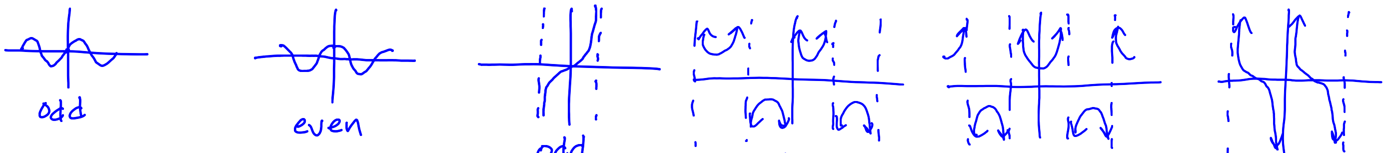
Even functions have graphs that are symmetric with respect to the y-axis $(y=x^2)$



In odd functions, $f(-x) = -f(x)$. In even functions, $f(-x) = f(x)$.

Sketch the graphs of the six parent functions below, and decide which are odd and which are even.

$f(x) = \sin x$ $\quad$ $f(x) = \cos x$ $\quad$ $f(x) = \tan x$ $\quad$ $f(x) = \csc x$ $\quad$ $f(x) = \sec x$ $\quad$ $f(x) = \cot x$



odd $\quad$ even $\quad$ odd $\quad$ odd $\quad$ even $\quad$ odd

Fill in the blanks to complete the odd and even identities:

even: $\cos(-x) = \cos x$ $\qquad$ $\sec(-x) = \sec x$

odd: $\sin(-x) = -\sin x$ $\qquad$ $\tan(-x) = -\tan x$ $\qquad$ $\csc(-x) = -\csc x$ $\qquad$ $\cot(-x) = -\cot x$

Examples: Simplify the following.

a) $\csc(-x)\tan(-x)$

$$= (-\csc x)(-\tan x)$$
$$= \csc x \tan x$$
$$= \left(\frac{1}{\cancel{\sin x}}\right)\left(\frac{\cancel{\sin x}}{\cos x}\right)$$
$$= \frac{1}{\cos x} = \boxed{\sec x}$$

b) $\dfrac{1}{1+\cos(-x)} + \dfrac{1}{1-\cos x}$

$$= \frac{1}{1+\cos x} + \frac{1}{1-\cos x}$$
$$= \left(\frac{1-\cos x}{1-\cos x}\right)\left(\frac{1}{1+\cos x}\right) + \left(\frac{1}{1-\cos x}\right)\left(\frac{1+\cos x}{1+\cos x}\right)$$
$$= \frac{1-\cos x + 1 + \cos x}{(1-\cos x)(1+\cos x)}$$
$$= \frac{2}{1-\cos^2 x}$$
$$= \frac{2}{\sin^2 x}$$
$$= 2\left(\frac{1}{\sin^2 x}\right) = \boxed{2\csc^2 x}$$

c) $\tan^2(-x) - \dfrac{\csc^2 x}{\cot^2 x}$

$$= (-\tan x)^2 - \frac{\csc^2 x}{\cot^2 x}$$
$$= \tan^2 x - \frac{\csc^2 x}{\cot^2 x}$$
$$= \frac{\sin^2 x}{\cos^2 x} - \frac{\frac{1}{\sin^2 x}}{\frac{\cos^2 x}{\sin^2 x}}$$
$$= \frac{\sin^2 x}{\cos^2 x} - \left(\frac{1}{\cancel{\sin^2 x}}\right)\left(\frac{\cancel{\sin^2 x}}{\cos^2 x}\right)$$
$$= \frac{\sin^2 x}{\cos^2 x} - \frac{1}{\cos^2 x}$$
$$= \frac{\sin^2 x - 1}{\cos^2 x} = \frac{-\cos^2 x}{\cos^2 x} = \boxed{-1}$$