Derivatives of Exponential and Logarithmic Functions

When we reviewed exponentials and logarithms in chapter 1 we mentioned that logs and exponential with base *e* come up a great deal. *e* is extremely unique, not just because it shows up in nature; but for several other reasons. One we will examine now.

Recall $\lim_{x\to 0}\frac{\sin x}{x}=1$ we first showed this using a graph.

Consider $\lim_{h\to 0}\frac{e^h-1}{h}=1$ *Graph*

This means something remarkable when we find the derivative of e^x.

$$\frac{d}{dx}(e^x)=\lim_{h\to 0}\frac{e^{x+h}-e^x}{h}=\frac{e^x\cdot e^h-e^x}{h}=\lim_{h\to 0}\frac{e^x(e^h-1)}{h}$$

$$=\lim_{h\to 0}e^x\,\frac{(e^h-1)}{h}=\lim_{h\to 0}e^x\cdot\lim_{h\to 0}\frac{(e^h-1)}{h}=e^x\cdot 1=e^x$$

$$\frac{d}{dx}(e^x)=e^x$$

The function is its own derivative!
In a general form we apply the chain rule and get

$$\frac{d}{dx}e^u=e^u\frac{du}{dx}$$

It would be nice to differentiate other exponential functions. Let's try a clever trick. Knowing

$$\frac{d}{dx}(e^x)=e^x \quad and \quad a^x=e^{x\ln a}$$

$$\frac{d}{dx}e^{x\ln a}=e^{x\ln a}(\ln a)=e^{\ln a^x}\cdot\ln a=a^x\ln a$$

$$\frac{d}{dx}a^x=a^x\ln a$$

$$In\ general:\ \frac{d}{dx}a^u=a^u\ln a\frac{du}{dx}$$

Examples:

$y=2e^x$ $\qquad \frac{d}{dx}=2e^x$

$y=e^{2x}$ $\qquad \frac{d}{dx}=e^{2x}\cdot 2 \qquad \frac{d}{dx}=2e^{2x}$

$y=8^x$ $\qquad y'=8^x\ln 8$

$y=x^\pi$ $\qquad y'=\pi(x)^{\pi-1}$

$y=3^{\csc(x)}$ $\qquad y'=-3^{\csc(x)}\ln 3(\csc x\cot x)$

That takes care of exponentials, not that they are all that easy. What about logarithms. Let's do another clever move.

$y = \ln(x) \qquad e^y = e^{\ln(x)} \qquad e^y = x$

$$\frac{d}{dx}e^y = \frac{d}{dx}x$$

$$e^y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

$$so \ \frac{d}{dx}\ln(x) = \frac{1}{x}$$

In general: $\frac{d}{dx}\ln(u) = \frac{1}{u}\frac{du}{dx}$

Derive a formula for $\frac{d}{dx}\log_a x$

How could this be rewritten with functions you already know?
Use Change of base formula:

$$\frac{d}{dx}\frac{\ln x}{\ln a} = \frac{1}{\ln a}\frac{d}{dx}\ln x = \frac{1}{\ln a}\cdot\frac{1}{x} = \frac{1}{x\ln a}$$

$$In \ \ general \ \ \frac{d}{dx}\log_a u = \frac{1}{u\ln a}\frac{du}{dx}$$

Examples:

15. $y = \ln(x^2) \qquad y' = \frac{1}{x^2}2x = \frac{2}{x}$

16. $y = \left(\ln x\right)^2 \qquad y' = 2\ln x\frac{1}{x}$

21. $y = \log_4 x^2 \qquad y' = \frac{1}{x^2\ln 4}2x = \frac{2}{x\ln 4}$

24. $y = \frac{1}{\log_2 x}$

$$y' = (\log_2 x)^{-1} = -1(\log_2 x)^{-2}\frac{1}{x\ln 2} = \frac{-1}{(\log_2 x)^2 x\ln 2}$$

$$= \frac{-1}{x(\ln 2)(\log_2 x)^2} = \frac{-1\ln 2}{x(\ln x)^2}$$

Exploration 1

Example 8

We need to revisit the power rule one more time

If u is a positive differentiable function and n is any real number then

$$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}$$

Example 5

Example 6

Example 7

Logarithmic Differentiation : do when both base and exponent are functions of x.

#47. $y = x^{\ln x}$