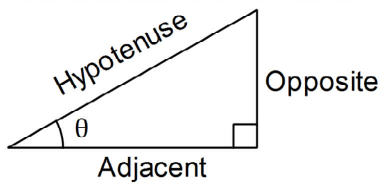
Circular Trigonometry

The six trigonometric functions are the sine (sin), cosine (cos), tangent (tan), cosecant (csc), secant (sec), and cotangent (cot) functions. We saw how to define sine, cosine, and tangent in a right triangle. Cosecant, secant, and cotangent are their reciprocals.

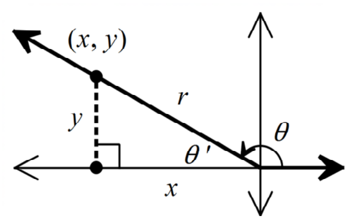
Reciprocal Identities: $\csc\theta = \dfrac{1}{\sin\theta}$ $\sec\theta = \dfrac{1}{\cos\theta}$ $\cot\theta = \dfrac{1}{\tan\theta}$



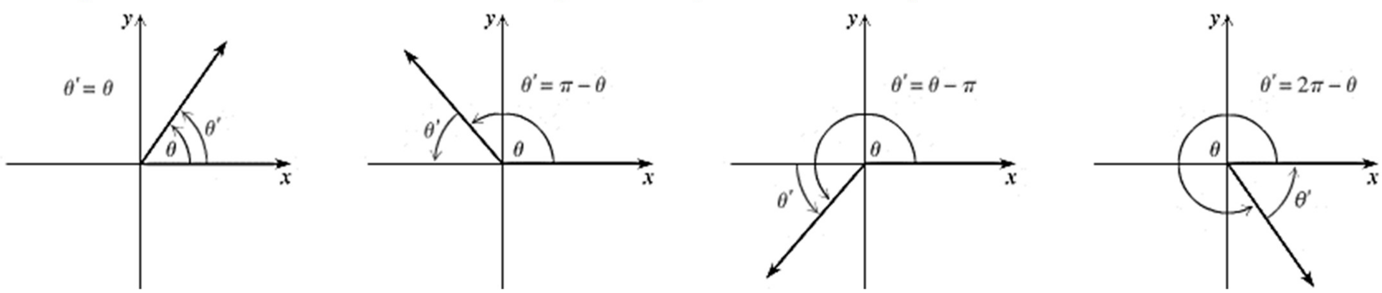
$\sin\theta = \dfrac{\text{opp}}{\text{hyp}}$ $\cos\theta = \dfrac{\text{adj}}{\text{hyp}}$ $\tan\theta = \dfrac{\text{opp}}{\text{adj}}$

$\csc\theta = \dfrac{\text{hyp}}{\text{opp}}$ $\sec\theta = \dfrac{\text{hyp}}{\text{adj}}$ $\cot\theta = \dfrac{\text{adj}}{\text{opp}}$

We can extend the definitions of the trigonometric functions so they apply to angles with *any* measure, rather than just acute angles in a right triangle. If (x, y) is a point on the terminal side of an angle θ in standard position, we can draw a triangle by drawing a segment straight up or down from the point to the x-axis. The triangle has legs x and y (these may be positive or negative) and hypotenuse r, where $r = \sqrt{x^2 + y^2}$. If θ is an angle in standard position, then the ***reference angle*** θ' (read "theta prime") is the angle between the terminal side of the angle and the positive or negative x-axis.



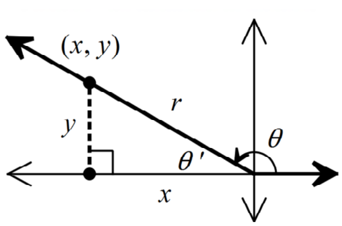
The diagram below shows what reference angles for angles in different quadrants look like.



The technical definitions of the six trigonometric functions involve x, y, and r, but if you want to think in terms of SOH-CAH-TOA, draw a triangle and look at the legs opposite and adjacent to the reference angle.

Definitions of Trigonometric Functions in the Coordinate Plane

If (x, y) is a point on the terminal side of an angle θ in standard position and $r = \sqrt{x^2 + y^2}$, then:



$\sin\theta = \dfrac{\text{opp}}{\text{hyp}} = \dfrac{y}{r}$ $\cos\theta = \dfrac{\text{adj}}{\text{hyp}} = \dfrac{x}{r}$ $\tan\theta = \dfrac{\text{opp}}{\text{adj}} = \dfrac{y}{x}$

$\csc\theta = \dfrac{\text{hyp}}{\text{opp}} = \dfrac{r}{y}$ $\sec\theta = \dfrac{\text{hyp}}{\text{adj}} = \dfrac{r}{x}$ $\cot\theta = \dfrac{\text{adj}}{\text{opp}} = \dfrac{x}{y}$

The *signs* of the trigonometric functions depend on the quadrant in which the angle lies and the corresponding signs of x and y (remember r is always positive).

Quadrant II: x: ______ y: ______ r: ______	Quadrant I: x: ______ y: ______ r: ______
$\sin\theta = y/r$ and $\csc\theta = r/y$: _______	$\sin\theta = y/r$ and $\csc\theta = r/y$: _______
$\cos\theta = x/r$ and $\sec\theta = r/x$: _______	$\cos\theta = x/r$ and $\sec\theta = r/x$: _______
$\tan\theta = y/x$ and $\cot\theta = x/y$: _______	$\tan\theta = y/x$ and $\cot\theta = x/y$: _______
Quadrant III: x: ______ y: ______ r: ______	Quadrant IV: x: ______ y: ______ r: ______
$\sin\theta = y/r$ and $\csc\theta = r/y$: _______	$\sin\theta = y/r$ and $\csc\theta = r/y$: _______
$\cos\theta = x/r$ and $\sec\theta = r/x$: _______	$\cos\theta = x/r$ and $\sec\theta = r/x$: _______
$\tan\theta = y/x$ and $\cot\theta = x/y$: _______	$\tan\theta = y/x$ and $\cot\theta = x/y$: _______

Examples:

Find the values of the six trigonometric functions of the angle α in standard position whose terminal side passes through $(-2,-4)$.

Find the values of the other five trigonometric functions based on the quadrant of the angle and the value of the given function.

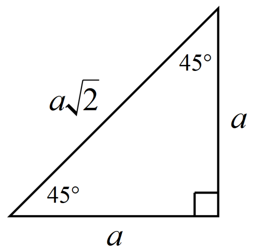
a) $\cos\alpha = \frac{1}{4}$, α is in Quadrant I

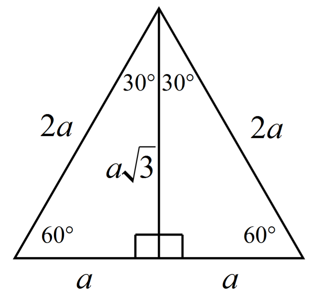
b) $\tan\beta = -\frac{12}{5}$, β is in Quadrant II

c) $\sin\theta = -\frac{\sqrt{5}}{3}$, θ is in Quadrant III.

d) $\sec\theta = \frac{7}{3}$, θ is in Quadrant IV

We can find the use the ratios that exist in special right triangles to calculate the coordinates of points on the unit circle.



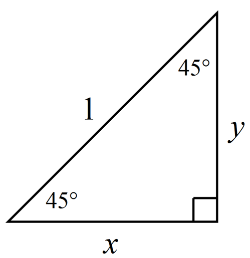


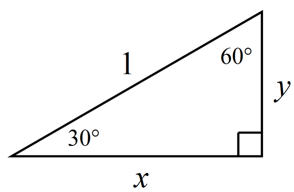
hypotenuse = leg $\cdot \sqrt{2}$

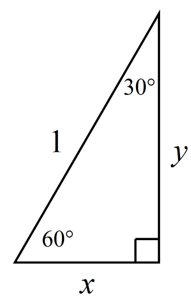
hypotenuse = 2 $\cdot$ short leg

long leg = short leg $\cdot \sqrt{3}$

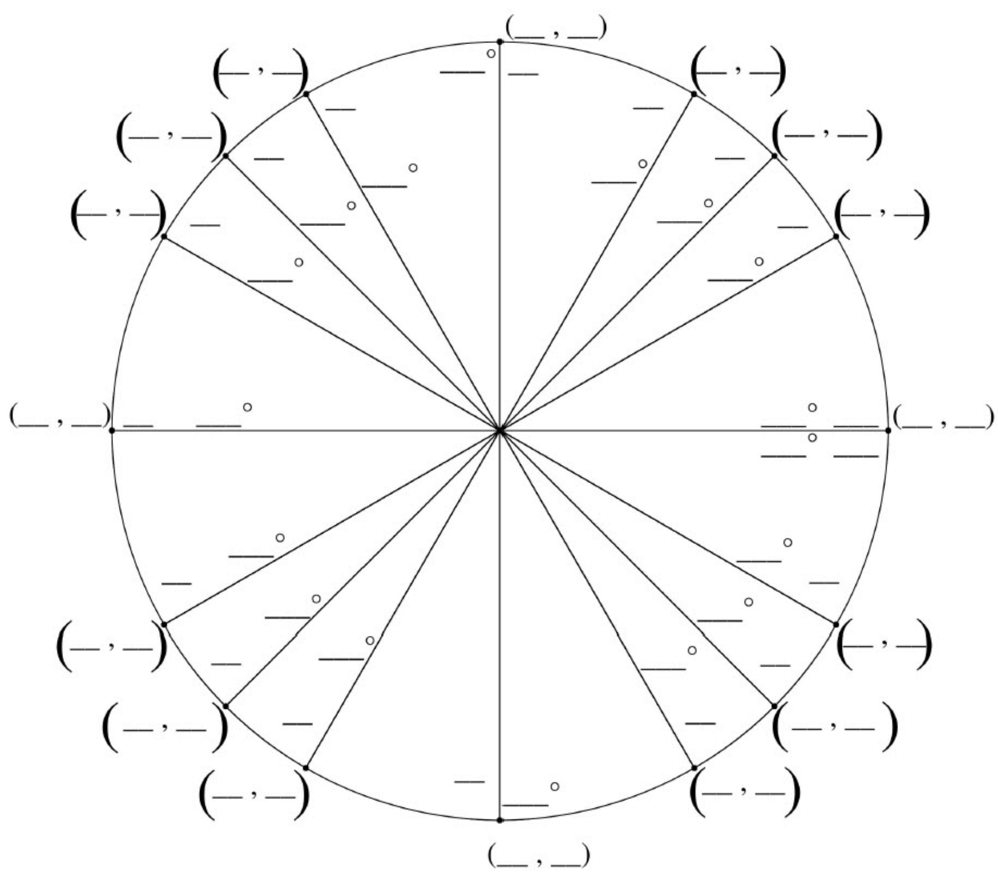
Figure out the values of x and y in the triangles below:







Examples: Use special right triangles to fill in the x and y-coordinates for the main angles on the unit circle.



Since the radius of the unit circle is 1, the values of the trigonometric functions for any unit circle angle can be calculated easily from looking at the coordinates.

On the Unit Circle $(r = 1)$:

$$\sin\theta = \frac{y}{r} = y \qquad \cos\theta = \frac{x}{r} = x \qquad \tan\theta = \frac{y}{x}$$

$$\csc\theta = \frac{r}{y} = \frac{1}{y} \qquad \sec\theta = \frac{r}{x} = \frac{1}{x} \qquad \cot\theta = \frac{x}{y}$$

Notes:

- ★ $\frac{2}{\sqrt{2}} = \frac{2}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{2\sqrt{2}}{2} = \sqrt{2}$. (This makes sense because $\frac{2}{\sqrt{2}}$ is the reciprocal of $\frac{\sqrt{2}}{2}$, and $\frac{\sqrt{2}}{2}$ is just another way of writing $\frac{1}{\sqrt{2}}$, which has a reciprocal of $\sqrt{2}$.) Please do not write any answers as $\frac{2}{\sqrt{2}}$. You need to know that $\frac{2}{\sqrt{2}} = \sqrt{2}$.
- ★ $\frac{1/2}{\sqrt{3}/2} = \frac{1}{2} \cdot \frac{2}{\sqrt{3}} = \frac{1}{\sqrt{3}}$. This answer can also be written as $\frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$. Know that $\frac{1}{\sqrt{3}}$ and $\frac{\sqrt{3}}{3}$ are the same number. Either form is acceptable. The same is true for $\frac{2}{\sqrt{3}}$ and $\frac{2\sqrt{3}}{3}$.

Examples: Find the exact values of the following:

1. $\sin 0°$	2. $\cos \pi$	3. $\tan(-\pi/2)$	4. $\csc(-270°)$
5. $\sin(\pi/4)$	6. $\cos(-225°)$	7. $\cot(13\pi/4)$	8. $\sec 315°$
9. $\sin 30°$	10. $\cos(7\pi/6)$	11. $\tan(-\pi/3)$	12. $\csc 150°$
13. $\cot(-240°)$	14. $\sec(-\pi/6)$	15. $\cos(5\pi/3)$	16. $\tan(-150°)$

Inverse Sine, Cosine, and Tangent Functions

$\sin^{-1} x$ or $\arcsin x$ is the angle in $[-90°, 90°]$ or $[-\frac{\pi}{2}, \frac{\pi}{2}]$ whose sine is x.
$\cos^{-1} x$ or $\arccos x$ is the angle in $[0°, 180°]$ or $[0, \pi]$ whose cosine is x.
$\tan^{-1} x$ or $\arctan x$ is the angle in $[-90°, 90°]$ or $[-\frac{\pi}{2}, \frac{\pi}{2}]$ whose tangent is x.

Examples: Find the exact values of the following, in radians.

1. $\arcsin\left(\frac{\sqrt{3}}{2}\right)$
2. $\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$
3. $\arctan\left(\sqrt{3}\right)$
4. $\sin^{-1}\left(-\frac{1}{2}\right)$
5. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$
6. $\arctan\left(-\frac{\sqrt{3}}{3}\right)$
7. $\arccos(-1)$
8. $\tan^{-1}(-1)$
9. $\arcsin(0)$
10. $\arccos\left(\frac{1}{2}\right)$
11. $\sin^{-1}(1)$
12. $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$