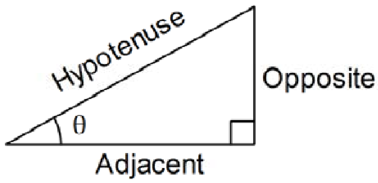
Right Triangle Trigonometry

Trigonometric Ratio: A ratio of the lengths of two sides of a right triangle. The three main trigonometric rations are sine (sin), cosine (cos), and tangent (tan).

★ If   is an acute angle of a right triangle, then the values of the sine, cosine, and tangent of   are defined in terms of the lengths of the three sides of the triangle – the leg adjacent (next to) , the leg opposite (across from) , and the hypotenuse – as follows:



$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

A common way to remember this is SOH-CAH-TOA

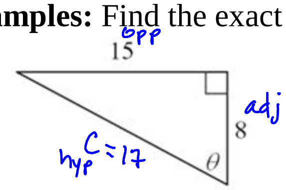
★ **Remember:** For all right triangles, you can use the Pythagorean Theorem, $a^2 + b^2 = c^2$.

$leg^2 + leg^2 = hyp^2$

$hyp = \sqrt{leg^2 + leg^2}$ or $leg = \sqrt{hyp^2 - leg^2}$

Examples: Find the exact values of $\sin\theta$, $\cos\theta$, and $\tan\theta$.

a)



$8^2 + 15^2 = c^2$

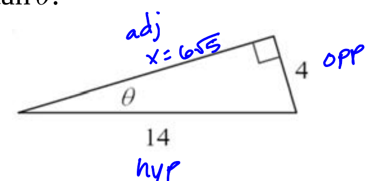
$c = \sqrt{8^2 + 15^2} = \sqrt{289} = 17$

$\sin\theta = \boxed{\frac{15}{17}}$

$\cos\theta = \boxed{\frac{8}{17}}$

$\tan\theta = \boxed{\frac{15}{8}}$

b)



$x^2 + 4^2 = 14^2$

$x^2 = 14^2 - 4^2$

$x = \sqrt{14^2 - 4^2} = \sqrt{180}$

$180 = 18 \cdot 10 = 9 \cdot 2 \cdot 2 \cdot 5 = 3 \cdot 3 \cdot 2 \cdot 2 \cdot 5$

$\sqrt{3 \cdot 3} \cdot \sqrt{2 \cdot 2} \cdot \sqrt{5} = 3 \cdot 2 \cdot \sqrt{5} = 6\sqrt{5}$

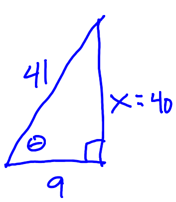
$\sin\theta = \frac{4}{14} = \boxed{\frac{2}{7}}$

$\cos\theta = \frac{6\sqrt{5}}{14} = \boxed{\frac{3\sqrt{5}}{7}}$

$\tan\theta = \frac{4}{6\sqrt{5}} = \boxed{\frac{2}{3\sqrt{5}}}$

Examples: Draw and label a triangle, find the length of the missing side, and find the requested values.

a) Find $\sin\theta$ and $\tan\theta$ if $\cos\theta = \frac{9}{41}$ $\frac{adj}{hyp}$

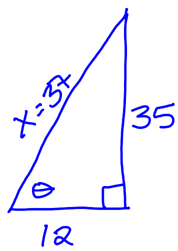


$x = \sqrt{41^2 - 9^2} = 40$

$\boxed{\sin\theta = \frac{40}{41}}$

$\boxed{\tan\theta = \frac{40}{9}}$

b) Find $\sin\theta$ and $\cos\theta$ if $\tan\theta = \frac{35}{12}$ $\frac{opp}{adj}$

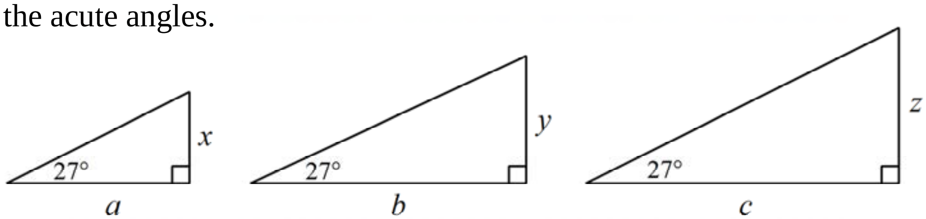


$x = \sqrt{12^2 + 35^2} = 37$

$\boxed{\sin\theta = \frac{35}{37}}$

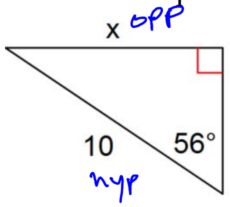
$\boxed{\cos\theta = \frac{12}{37}}$

No matter how big the triangle is, the values of the trigonometric functions for a certain size angle will remain the same. For example, in the diagram below, $\tan 27° = x/a = y/b = z/c$. The value of tan 27° never changes, no matter what size the triangle is. In fact, tan 27° ≈ 0.5095254495… Sine and cosine act in a similar way. The sine or cosine of a certain angle is the same no matter what size the triangle is. This allows us to use the value of trigonometric functions to find the length of a side of a right triangle if we know the length of another side and the size of one of the acute angles.



Examples: Write an equation involving sine, cosine, or tangent that can be used to find the missing length. Then solve the equation. Round your answers to the nearest tenth.

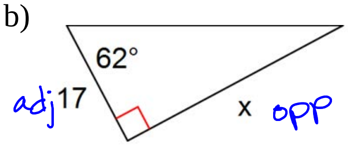
a)



$\sin 56° = \frac{x}{10}$

$x = 10 \sin 56° \approx \boxed{8.3}$

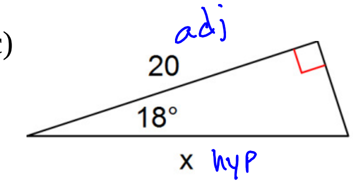
b)



$\tan 62° = \frac{x}{17}$

$x = 17 \tan 62° \approx \boxed{32.0}$

c)

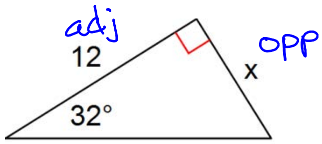


$\cos 18° = \frac{20}{x}$

$x \cos 18° = 20$

$x = \frac{20}{\cos 18°} \approx \boxed{21.0}$

d)

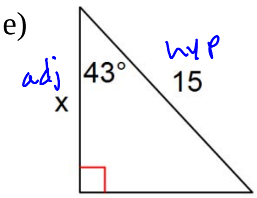


$\tan 32° = \frac{x}{12}$

$x = 12 \tan 32°$

$\approx \boxed{7.5}$

e)

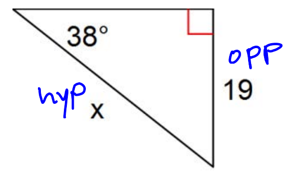


$\cos 43° = \frac{x}{15}$

$x = 15 \cos 43°$

$\approx \boxed{11.0}$

f)



$\sin 38° = \frac{19}{x}$

$x \sin 38° = 19$

$x = \frac{19}{\sin 38°} \approx \boxed{30.9}$

Inverse Sine, Cosine, and Tangent Functions

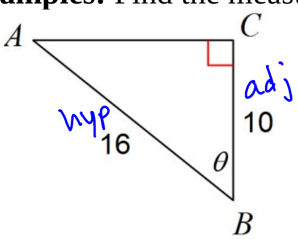
- The inverse sine or arcsine of x ($\sin^{-1} x$ or $\arcsin x$) is the angle between $-90°$ and $90°$ whose sine is x.
  - If $\sin\theta = x$, and $-90° \le \theta \le 90°$, then $\theta = \sin^{-1} x$.
- The inverse cosine or arccosine of x ($\cos^{-1} x$ or $\arccos x$) is the angle between $0°$ and $180°$ whose cosine is x.
  - If $\cos\theta = x$, and $0° \le \theta \le 180°$, then $\theta = \cos^{-1} x$.
- The inverse tangent or arctangent of x ($\tan^{-1} x$ or $\arctan x$) is the angle between $-90°$ and $90°$ whose tangent is x.
  - If $\tan\theta = x$, and $-90° < \theta < 90°$, then $\theta = \tan^{-1} x$.

★ **The -1 in $\sin^{-1} x$ does not indicate a reciprocal! $\sin^{-1} x \ne 1/\sin x$.**
$\sin^{-1} x$, $\cos^{-1} x$, and $\tan^{-1} x$ are angles!

★ **Use inverse functions when you know the sine, cosine, or tangent of an angle and want to know how big the angle is.**

Examples: Find the measure of the indicated angle to the nearest tenth of a degree.

a)



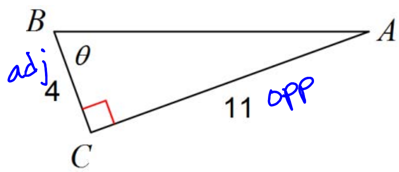
$\cos\theta = \frac{10}{16}$

$\theta = \cos^{-1}\left(\frac{10}{16}\right) \approx \boxed{51.3°}$

DON'T WRITE $\cos^{-1}(\theta)$!

You don't take $\cos^{-1}$ of an angle. The angle is the $\cos^{-1}$.

b)

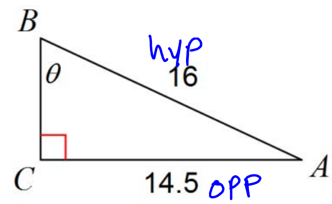


$\tan\theta = \frac{4}{4}$

$\theta = \tan^{-1}\left(\frac{11}{4}\right)$

$\theta \approx \boxed{70.0°}$

c)



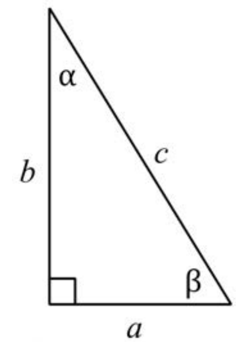
$\sin\theta = \frac{14.5}{16}$

$\theta = \sin^{-1}\left(\frac{14.5}{16}\right)$

$\boxed{\theta \approx 65.0°}$

Solving Right Triangles

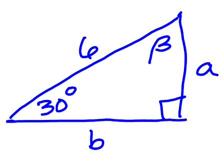
Finding all the missing angle measures and side lengths of a triangle is called “solving a triangle”. In a right triangle, we often name the acute angles α (alpha) and β (beta) and the lengths of the sides opposite those angles a and b, respectively. The right angle is sometimes called γ (gamma) and the length of the side opposite the right angle (the hypotenuse) is c.



- ★ If you know the lengths of two of the sides and want the length of the third side, use the Pythagorean Theorem.
- ★ If you know the measure of one of the acute angles and want the measure of the other acute angle, use the fact that the angles in a triangle add to 180° (so the acute angles in a right triangle add to 90°).
- ★ If you know the measure of one angle and the length of one side and want the lengths of the other sides, use sin, cos, or tan.
- ★ If you know the lengths of the sides and want to figure out the angle measures, use inverse functions ($\sin^{-1}$, $\cos^{-1}$, or $\tan^{-1}$).

Examples:

Solve the right triangle in which $\alpha = 30°$ and $c = 6$.



$\beta = 90° - 30° = \boxed{60°}$

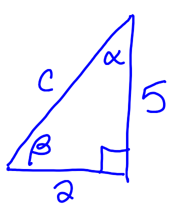
$\sin 30° = \frac{a}{6}$

$a = 6 \sin 30° = \boxed{3}$

$\cos 30° = \frac{b}{6}$

$b = 6 \cos 30° \approx \boxed{5.2}$

Solve the right triangle in which $a = 2$ and $b = 5$.

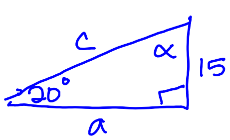


$c = \sqrt{2^2 + 5^2} = \sqrt{29} \approx \boxed{5.4}$

$\tan \alpha = \frac{2}{5}$ $\alpha = \tan^{-1}\left(\frac{2}{5}\right) \approx \boxed{21.8°}$

$\tan \beta = \frac{5}{2}$ $\beta = \tan^{-1}\left(\frac{5}{2}\right) \approx \boxed{68.2°}$

Solve the right triangle in which $\beta = 20°$ and $b = 15$.

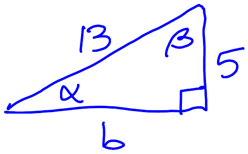


$\alpha = 90° - 20° = \boxed{70°}$

$\tan 20° = \frac{15}{a}$ $a = \frac{15}{\tan 20°} \approx \boxed{41.2}$

$\sin 20° = \frac{15}{c}$ $c = \frac{15}{\sin 20°} \approx \boxed{43.9}$

Solve the right triangle in which $a = 5$ and $c = 13$.

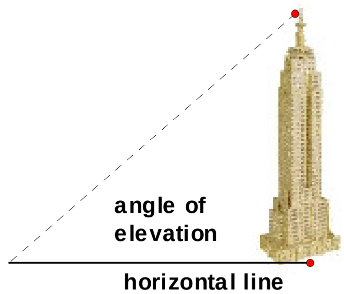


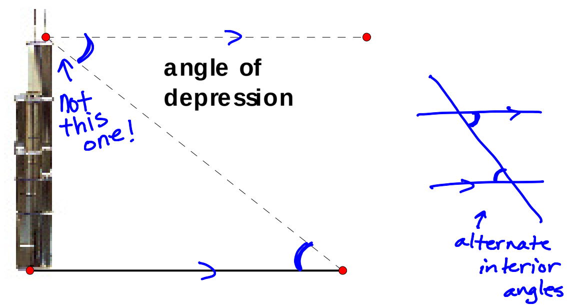
$b = \sqrt{13^2 - 5^2} = \boxed{12}$

$\sin \alpha = \frac{5}{13}$ $\alpha = \sin^{-1}\left(\frac{5}{13}\right) \approx \boxed{22.6°}$

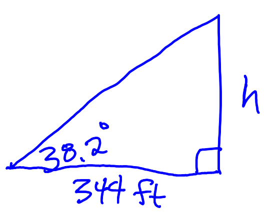
$\cos \beta = \frac{5}{13}$ $\beta = \cos^{-1}\left(\frac{5}{13}\right) = \boxed{67.4°}$

Using trigonometry, we can find the size of an object without actually measuring the object. Two common terms used in this regard are **angle of elevation** and **angle of depression**.





Examples:
The angle of elevation of the top of a cell phone tower is 38.2° at a distance of 344 feet from the tower. What is the height of the tower?

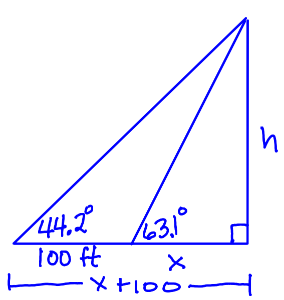


$$\tan 38.2^\circ = \frac{h}{344}$$

$$h = 344 \tan 38.2^\circ$$

$$h \approx \boxed{270.7 \text{ ft}}$$

At one location, the angle of elevation of the top of an antenna is 44.2°. At a point that is 100 feet closer to the antenna, the angle of elevation is 63.1°. What is the height of the antenna?



$$\tan 63.1^\circ = \frac{h}{x}$$

$$h = x \tan 63.1^\circ$$

$$\tan 44.2^\circ = \frac{h}{x+100}$$

$$(x+100)(\tan 44.2^\circ) = \left(\frac{x \tan 63.1^\circ}{\cancel{x+100}}\right)(\cancel{x+100})$$

$$\cancel{x \tan 44.2^\circ} + 100 \tan 44.2^\circ = x \tan 63.1^\circ$$
$$\cancel{-x \tan 44.2^\circ} \qquad\qquad -x \tan 44.2^\circ$$

$$100 \tan 44.2^\circ = x \tan 63.1^\circ - x \tan 44.2^\circ$$

$$\frac{100 \tan 44.2^\circ}{\tan 63.1^\circ - \tan 44.2^\circ} = \frac{x\,\cancel{(\tan 63.1^\circ - \tan 44.2^\circ)}}{\cancel{\tan 63.1^\circ - \tan 44.2^\circ}}$$

$$x = \frac{100 \tan 44.2^\circ}{\tan 63.1^\circ - \tan 44.2^\circ} \approx 97.4 \text{ ft}$$

$$h = x \tan 63.1^\circ$$

$$h = 97.4 \tan 63.1^\circ$$

$$\boxed{h \approx 191.9 \text{ ft}}$$