Derivatives of Inverse Trigonometric Functions

Recall from our chapter 1 review that the inverse of a function can be obtained by switching x and y and solving for y. An inverse is a reflection of the function over the line y = x. Consider Figure 4.12 (The graphs of a function and its inverse; notice that the tangent lines have reciprocal slopes.)

Think of this….the slope of the function y is the reciprocal of the slope of its inverse, because you switch the x and y…..
So it is with derivatives….

The derivative idea should make sense because if we switched x and y to get y^{-1} then dy should be dx and dx → dy. $\frac{dy^{-1}}{dx}=\frac{dx}{dy}$

Example: if

$$f(\text{a})=\text{b} \quad and \quad f'(a)=c \quad and\, if \quad g(x)=f^{-1}(x), then\, g'(b)=\frac{1}{c}$$

$$Example: f(2)=5 \quad and \quad f'(2)=10 \quad and\, if \quad g(x)=f^{-1}(x), then\, g'(5)=\frac{1}{10}$$

Notice further that if the function is smooth and continuous f^{-1} will be smooth and continuous. We do have to be concerned with the domain!

Example: f(x) = sin (x)

$$f^{-1}(x)=\sin^{-1}x \quad \text{Restricted domain } [-\frac{\pi}{2},\frac{\pi}{2}]$$

Consider $y = \sin^{-1}(x)$ what is dy/dx?
Graph the function. It has a vertical tangent, but it should be differentiable everywhere else.

$$Find\ \frac{dy}{dx} \qquad if \qquad y=\sin^{-1}(x)$$

$$\sin y = x \qquad \cos y \bullet \frac{dy}{dx}=1$$

$$\frac{dy}{dx}=\frac{1}{\cos y}$$

We need this to be a function of x, not y.
We now do something very clever.

$$\cos^2 y = 1-\sin^2 y \Rightarrow \cos y=\sqrt{1-\sin^2 y}$$

Sin(y) = x coordinate

So $\cos y=\sqrt{1-x^2}$ We can do this because cos y is always positive in the domain

$$\frac{d}{dx}\sin^{-1}(x)=\frac{1}{\sqrt{1-x^2}}$$

If u is a differentiable function of x, we apply the chain rule and get

$$\frac{d}{dx}\sin^{-1}u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \qquad |u|<1$$

Examples: $y = \sin^{-1}(2x) \quad Find\ \frac{dy}{dx}$ $\qquad y = \sin^{-1}(1-t) \quad Find\ \frac{dy}{dt}$

$$\frac{d}{dx}\tan^{-1}u = \frac{1}{1+u^2}\frac{du}{dx}$$

$$\frac{d}{dx}\sec^{-1}u = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx} \qquad |u|>1$$

The derivatives of the cofunctions are negatives of one another.
Inverse Function – Inverse Cofunction Identities
Given the cofunctions find the derivatives.

$$\cos^{-1}x = \frac{\pi}{2} - \sin^{-1}x \qquad \frac{d}{dx}\cos^{-1}x =$$

$$\cot^{-1}x = \frac{\pi}{2} - \tan^{-1}x \qquad \frac{d}{dx}\cot^{-1}x =$$

$$\csc^{-1}x = \frac{\pi}{2} - \sec^{-1}x \qquad \frac{d}{dx}\csc^{-1}x =$$

Calculator Conversion Identities

$$\sec^{-1}x = \cos^{-1}(1/x)$$

$$\cot^{-1}x = \frac{\pi}{2} - \tan^{-1}(x)$$

$$\csc^{-1}x = \sin^{-1}(1/x)$$

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$$y = \cos^{-1}(x^2)$$

cofunction with $\sin^{-1}$

$$\frac{d}{dx}\sin^{-1}u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \quad so \quad \frac{d}{dx}\cos^{-1}u = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-(x^2)^2}}2x = \frac{-2x}{\sqrt{1-x^4}} \qquad |x|<1$$