Chain Rule

We know how to differentiate lots of functions, polynomials, trig functions, rational functions, etc., but we have not done derivatives of composites.

How do we differentiate $y=\sin(x^2+x)$?

We actually use a new rule for differentiation which is the ***most widely used rule in calculus***, the ***chain rule.***

Suppose we make an easy composite function. $y= 3(x^2+4x)$
This could be made up of $y=3(u)$ and $u = x^2+4x$ so $y = 3x^2+12x$

$\frac{dy}{dx} = 6x + 12$ $\quad\quad$ $\frac{dy}{du} = 3$ $\quad\quad$ $\frac{du}{dx} = 2x + 4$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

Here is another example:
$y = 9x^4 + 6x^2 + 1$
$y = u^2$ and $u = 3x^2 + 1$
$dy/du = 2u$ and $du/dx = 6x$
so $y' = 2(3x^2 + 1) \cdot 6x = (6x^2 + 2) \cdot 6x = 36x^3 + 12x$ Does this match with the derivative of $9x^4 + 6x^2 + 1$?

How can we write this as a rule that will be easier to work with? As it turns out, the previous rule is actually the notation that Leibniz worked out, but we generally use a method that Newton developed that relates to typical composite notation.

The Chain Rule:

If f is differentiable at the point $u = g(x)$ and g is differentiable at x, then the composite function
$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$

In Leibniz notation if $y = f(u)$ and $u = g(x)$, then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ where dy/du is evaluated at $u = g(x)$.

Remember this using the words “outside” and “inside”.
The derivative of the outside, leave the inside alone, times the derivative of the inside.

If $y = \sin(x^2+x)$ $\quad$ $f(x) = \sin(x)$ “outside” $\quad$ and $\quad$ $g(x) = x^2+x$ “inside”
Then y′ = derivative of the outside function evaluated at the inside function left alone times the derivative of the inside

$y = \sin(x^2+x)$

$y' = \cos(x^2+x) \cdot (2x + 1)$

Another example:
$y = 2(3x - 5)$ $\quad$ outside: $y = 2u$ $\quad\quad$ inside: $u= 3x - 5$

$y'= 2$ $\quad$ $u'= 3$ $\quad$ y′ of u times $u' = 2(3) = 6$. Does this match with the derivative of $6x - 10$?

After some practice, the rule becomes pretty easy to use.

Example Differentiate: sin(2x+1)

Outside derivative cos (2x + 1)
Inside derivative 2
Outside times inside: 2cos(2x + 1)

#3 $y = \cos(\sqrt{3} \cdot x)$

$\frac{d}{dx} outside \rightarrow -\sin(\sqrt{3} \cdot x)$ $\frac{d}{dx} inside \rightarrow \sqrt{3}$ $y' = -\sqrt{3}\sin(\sqrt{3} \cdot x)$

We can also apply the chain rule repeatedly
Consider: $y = \sin^4(3x) = (\sin(3x))^4 = 4(\sin(3x))^3 \cdot \cos(3x) \cdot 3 = 12\sin^3(3x)\cos(3x)$

#22 $y = (1+ \cos 2x)^2$
$2(1 + \cos 2x) (-\sin 2x) \cdot 2 = -4 (\sin 2x) (1+ \cos 2x)$

Example 3

Slopes of Parametrized Curves
Parametric curves still have tangents so they should have derivatives. Using the chain rule we can find dy/dx parametrically. If all three derivatives exist and dx/dt ≠ 0, then $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

Example: say x = sin t y = cos t $0 \le t \le 2\pi$
Find dy/dx.
Find the slope of the tangent at t = π/4.
Find the equation of the normal line at t = π/4.

Because powers are used so often and polynomials are so easy to differentiate, we have what's called "The Power Chain Rule".
It is easy to see by examples.

Example

$y = \sin^2(3x)$	$y = (x^3-2x)^4$	u^4
$2 \sin (3x) \cdot 3\cos(3x)$	$4(x^3 - 2x)^3 (3x^2 - 2)$	
$2u \cdot du/dx$	$4u^3 \cdot du/dx$	

If we think of these as $u^n \rightarrow d/dx = nu^{n-1} \cdot du/dx$

#23
$y = (1 + \cos^2 7x)^3$

$3 (1 + \cos^2 7x)^2(2\cos 7x)(-\sin 7x)7$

$-42 (\sin 7x) (\cos 7x) (1 + \cos^2 7x)^2$

Reminder, we always use θ measured in radians. All the formulas only work for radians. Yes it can be done in degrees, but it is very complicated!!!!!!