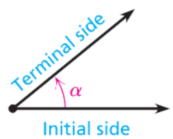
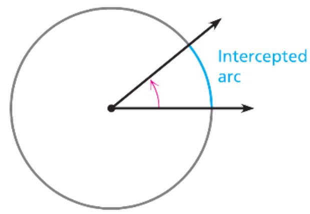
Angle Measures, Arc Length, and Sector Area

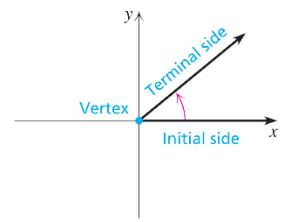
An **angle** can be formed by rotating one ray away from a fixed ray indicated by an arrow. The fixed ray is the **initial side** and the rotated ray is the **terminal side**. An angle whose vertex is the center of a circle is a **central angle**, and the arc of the circle through which the terminal side moves is the **intercepted arc**. An angle in **standard position** is located in a rectangular coordinate system with the vertex at the origin and the initial side on the positive *x*-axis.



Angle α



Central angle



Angle in standard position

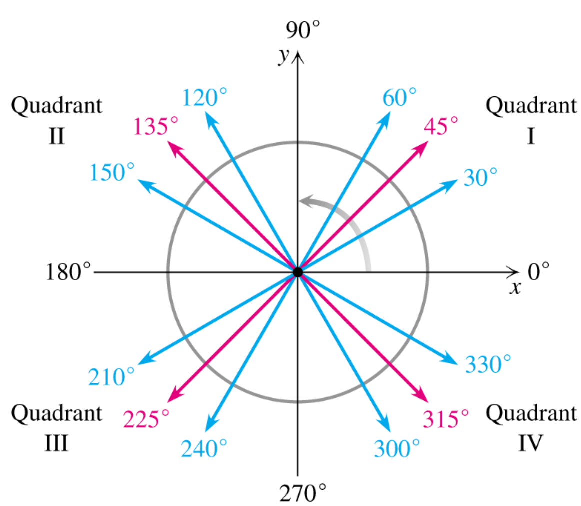
Degree Measure of Angles

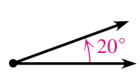
The measure, $m(\alpha)$, of an angle α is the amount of rotation from the initial side to the terminal side, and is found by using any circle centered at the vertex. An angle that forms a complete circle arc is 360°.

The **degree measure of an angle** is the number of degrees in the intercepted arc of a circle centered at the vertex.

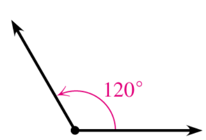
Counterclockwise rotation—positive angle **Clockwise rotation—negative angle**

An angle in standard position is said to lie in the quadrant where its terminal side lies.

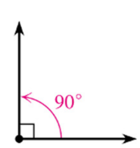




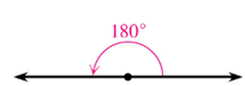
Acute angle



Obtuse angle



Right angle



Straight angle

Acute angle—An angle with a measure between 0° and 90°.
Obtuse angle—An angle with a measure between 90° and 180°.
Right angle—An angle with a measure of exactly 90°.
Straight angle—An angle with a measure of exactly 180°.

Example: Draw each angle in standard position, then name the quadrant in which the terminal side lies.

a) $255°$ b) $-650°$ c) $1360°$

Minutes and Seconds

Historically, angles were measured by using the **degrees-minutes-seconds (DMS) format**, but with calculators it is convenient to have some fractional parts of degrees written in decimal form. Each degree is divided into 60 equal parts called **minutes** (n'), and each minute is divided into 60 equal parts called **seconds** (n'').

1 degree = 60 minutes 1 minute = 60 seconds 1 degree = 3600 seconds

Examples: Convert each angle to decimal degrees. When necessary, round to four decimal places.

a) $18°24'$ b) $-10°15'42''$ c) $27°10'20''$

Examples: Convert each angle to degree-minutes-seconds format. Round to the nearest whole number of seconds.

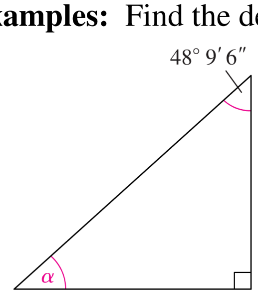
a) $56.6°$ b) $-17.45°$ c) $28.348°$

Examples: Perform the indicated operations.

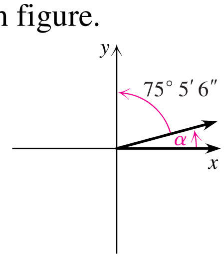
a) $15°56'45'' + 18°12'33''$ b) $36°5' - 22°33'12''$ c) $\dfrac{15°56'45''}{2}$

Examples: Find the degree measure of angle α in each figure.

a)



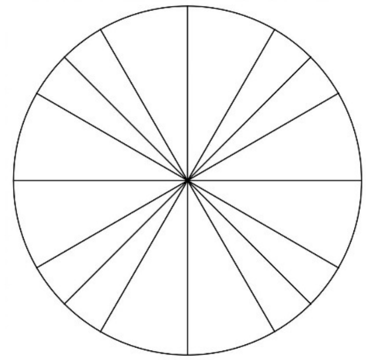
b)



Unit circle: A circle of radius one that is centered at the origin.

What is the circumference of the unit circle? (Circumference of a circle: $C = \pi d = 2\pi r$)

What is the arc length intercepted by a 180° angle (1/2 of the circle)?



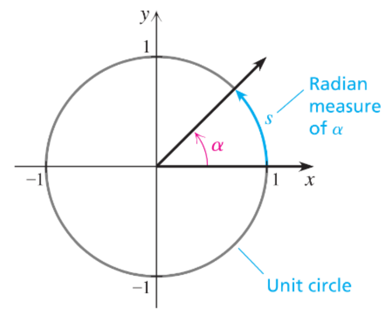
What is the arc length intercepted by a 120° angle (1/3 of the circle)?

What is the arc length intercepted by a 30° angle?

What is the arc length intercepted by a 225° angle?

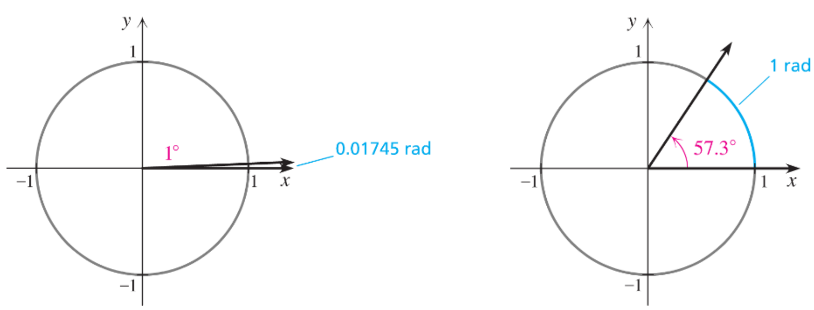
What is the arc length intercepted by a 210° angle?

You have just calculated the radian measure of each of these angles.



The **radian measure** of the angle α in standard position is the directed length of the intercepted arc on the unit circle. **Directed length** means that the radian measure is positive if the rotation of the terminal side is counterclockwise and negative if the rotation is clockwise.

One radian: The angle that intercepts an arc with length equal to the radius of a circle. (On the unit circle, one radian is the angle that intercepts an arc of length one.)



Since the radius of the unit circle is the real number 1, without any dimension (ft., meters, etc.), the arc length, and hence the radian measure of an angle, is also a real number without any dimension. Thus, one radian (1 rad) is the real number 1. If an equation asks you to find "all real number solutions" and the variable is an angle, it wants the solution(s) in radians. If an angle measure is given without a degree sign, assume it is in radians. ***If you write an angle measure and don't write a degree sign, I will assume you mean radians!***

Converting Between Radians and Degrees:

Since there are 2π radians in a circle (the circumference of the unit circle is 2π) and 360° in a circle,

$$2\pi \text{ radians} = 360°, \text{ or } \pi \text{ radians} = 180°.$$

Degrees→Radians: multiply by $\dfrac{\pi \text{ rad}}{180°}$.

Radians→Degrees: multiply by $\dfrac{180°}{\pi \text{ rad}}$.

Examples:

Convert the degree measures to radians:

a) 100° b) –27.2°

Convert the radian measures to degrees:

a) $-\dfrac{\pi}{12}$ b) 16.7 radians

Examples: Draw each angle in standard position and determine the quadrant in which each angle lies.

a) $-\dfrac{5\pi}{4}$ b) $\dfrac{10\pi}{3}$ c) 13.8 d) -2.5

The terminal side of an angle may be rotated in either a positive or negative direction to get to its final position. It can also be rotated for more than one revolution in either direction. Two angles with the same terminal side are called **coterminal angles.**

Coterminal Angles—Angles α and β are coterminal if and only if there is an integer k such that $m(\beta)=m(\alpha)+360°k$ or $m(\beta)=m(\alpha)+2\pi k.$ To find coterminal angles in degrees, add and subtract multiples of 360°. To find coterminal angles in radians, add or subtract multiples of 2π. Make sure to find a common denominator.

Examples: Find two positive angles and two negative angles that are coterminal with each angle:

a) 23° b) –146° c) $\dfrac{7\pi}{3}$ d) $-\dfrac{\pi}{4}$ e) 1.4

Examples: Determine whether the angles in each pair are coterminal:

a) –128° and 592° b) 8° and –368° c) $\dfrac{3\pi}{2}$ and $\dfrac{9\pi}{2}$

Arc Length

Often, we want to find the arc length (s) on a circle of radius r, intercepted by an angle α. We can do this by determining what fraction of the circle we are looking at, then multiplying by the circumference of the circle.

Degrees: $s = \dfrac{\alpha}{360°} \cdot 2\pi r$

Radians: $s = \dfrac{\alpha}{2\pi} \cdot 2\pi r$ or $\boldsymbol{s = \alpha r}$ **This formula only works if $\boldsymbol{\alpha}$ is in radians!**

Examples:

A central angle of $\pi/2$ intercepts an arc on the surface of the earth that runs from the equator to the North Pole. Using 3950 miles as the radius of the earth, find the length of the intercepted arc to the nearest mile.

A wagon wheel has a diameter of 28 inches and an angle of 30° between the spokes. What is the length of the arc s (to the nearest hundredth of an inch) between two adjacent spokes?

Area of a Sector of a Circle

Finding the area (A) of a sector of a circle of radius r with central angle α, is similar to finding the arc length:

Determine what fraction of the circle the sector makes up, then multiply by the area of the circle.

Degrees: $A = \dfrac{\alpha}{360°} \cdot \pi r^2$

Radians: $A = \dfrac{\alpha}{2\pi} \cdot \pi r^2$ or $\boldsymbol{A = \dfrac{\alpha r^2}{2}}$ **This formula only works if $\boldsymbol{\alpha}$ is in radians!**

Examples:

Which is bigger: a slice of pizza from a 10″ diameter pizza cut into 6 slices, or a slice from a 12″ diameter pizza cut into 8 slices?

A center-pivot irrigation system is used to water a circular field with radius 200 feet. In three hours the system waters a sector with a central angle of $\pi/8$. What area (in square feet) is watered in that time?