Compound Interest

***Simple Interest*:** If a principal of P dollars is borrowed for a period of t years at a per annum interest rate r, expressed as a decimal, the interest I charged is $I = Prt$.

When the interest due at the end of a payment period is added to the principal so that the interest computed at the end of the next payment period is based on the new principal amount (old principal plus interest), the interest is said to have been ***compounded***. ***Compound interest*** is interest paid on principal and previously earned interest.

Example: A credit union pays 7% per annum compounded quarterly on a certain savings plan. If \$900 is deposited in such a plan and the interest is left to accumulate, how much is in the account after 1 year? Hint: Find the simple interest each time the balance is compounded, adding the new interest each time.

For each quarter, $r = 0.07$, $t = 0.25$

Quarter 1:
$P = \$900$
$I = (\$900)(0.07)(0.25) = \$15.75$
$\$900 + \$15.75 = \$915.75$

Quarter 2:
$P = \$915.75$
$I = (\$915.75)(0.07)(0.25) = \$16.03$
$\$915.75 + \$16.03 = \$931.78$

Quarter 3:
$P = \$931.78$
$I = (\$931.78)(0.07)(0.25) = \$16.31$
$\$931.78 + \$16.31 = \$948.09$

Quarter 4:
$P = \$948.09$
$I = (\$948.09)(0.07)(0.25) = \$16.59$
$\$948.09 + \$16.59 = \boxed{\$964.68}$

Compound Interest Formula
The amount A after t years due to a principal P invested at an annual interest rate t compounded n times per year is $A = P \cdot \left(1 + \frac{r}{n}\right)^{nt}$.

Example: Investing \$1000 at an annual rate of 9% compounded annually, semiannually, quarterly, monthly, and daily will yield the following amounts after 3 years:

P $\quad r = 0.09 \quad t = 3$

Annual Compounding (n = 1):

$$A = 1000\left(1 + \frac{.09}{1}\right)^{(1)(3)} = \$1295.03$$

Semiannual Compounding (n = 2):

$$A = 1000\left(1 + \frac{.09}{2}\right)^{(2)(3)} = \$1302.26$$

Quarterly Compounding (n = 4):

$$A = 1000\left(1 + \frac{.09}{4}\right)^{(4)(3)} = \$1306.05$$

Monthly Compounding (n = 12):

$$A = 1000\left(1 + \frac{.09}{12}\right)^{(12)(3)} = \$1308.65$$

Daily Compounding (n = 365):

$$A = 1000\left(1 + \frac{.09}{365}\right)^{(365)(3)} = \$1309.92$$

Continuous Compounding

The amount A after t years due to a principal P invested at an annual interest rate r compounded continuously is $A = Pe^{rt}$.

Example: Find the amount A that results from investing a principle P of $1000 at an annual rate r of 9% compounded continuously for a time t of 3 years.

$$A = 1000e^{(0.09)(3)} = \$1309.96$$

Present Value: The amount of money that must be invested now in order to end up with a given amount after a certain amount of time.

Example: How much money must be invested now in order to end up with $20,000 ($A$) in 10 years ($t$) at

a) 5% compounded quarterly?

$r = 0.05$ $\quad n = 4$

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

$$20{,}000 = P\left(1+\frac{.05}{4}\right)^{(4)(10)}$$

$$P = \frac{20{,}000}{\left(1+\frac{0.05}{4}\right)^{(4)(10)}} \approx \boxed{\$12{,}168.27}$$

b) 3.8% compounded continuously?

$r = .038$ $\quad A = Pe^{rt}$

$$20{,}000 = Pe^{(0.038)(10)}$$

$$P = \frac{20{,}000}{e^{(0.038)(10)}} \approx \boxed{\$13{,}677.23}$$

Example: A zero-coupon (noninterest-bearing) bond can be redeemed in 9 years (t) for $1200 ($A$). How much should you be willing to pay for it now if you want a return of

a) 7% compounded monthly?

$r = 0.07$ $\quad n = 12$

$$1200 = P\left(1+\frac{0.07}{12}\right)^{(12)(9)}$$

$$P = \frac{1200}{\left(1+\frac{0.07}{12}\right)^{(12)(9)}} \approx \boxed{\$640.28}$$

b) 6% compounded continuously?

$r = 0.06$ $\quad A = Pe^{rt}$

$$1200 = Pe^{0.06(9)}$$

$$P = \frac{1200}{e^{(0.06)(9)}} \approx \boxed{\$699.30}$$

Example: What annual rate of interest compounded annually ($n=1$) should you seek if you want to double your investment ($A = 2P$) in 7 years (t)?

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

$$2P = P\left(1+\frac{r}{1}\right)^{(1)(7)}$$

$$\frac{2\cancel{P}}{\cancel{P}} = \frac{\cancel{P}(1+r)^7}{\cancel{P}}$$

$$2 = (1+r)^7$$

$$\sqrt[7]{2} = 1+r$$

$$r = \sqrt[7]{2} - 1 = 2^{1/7} - 1 \approx 0.104$$

$$\boxed{10.4\%}$$

Example: Find the time required to double ($A = 2P$) or triple ($A = 3P$) an investment if it earns 5% ($r = 0.05$) compounded continuously ($A = Pe^{rt}$).

Double: $A = Pe^{rt}$

$$\frac{2\cancel{P}}{\cancel{P}} = \frac{\cancel{P}e^{0.05t}}{\cancel{P}}$$

$$2 = e^{0.05t}$$

$$\ln 2 = 0.05t$$

$$t = \frac{\ln 2}{0.05} \approx \boxed{13.9 \text{ yrs}}$$

Triple: $A = Pe^{rt}$

$$\frac{3\cancel{P}}{\cancel{P}} = \frac{\cancel{P}e^{0.05t}}{\cancel{P}}$$

$$3 = e^{0.05t}$$

$$\ln 3 = 0.05t$$

$$t = \frac{\ln 3}{0.05} \approx \boxed{22.0 \text{ yrs}}$$

Exponential Growth and Decay

***Law of Uninhibited Growth or Decay*:**
Many natural phenomena have been found to follow the law that an amount $\boldsymbol{A}$ varies with time $\boldsymbol{t}$ according to the function $\boldsymbol{A}(\boldsymbol{t}) = \boldsymbol{A}_0\boldsymbol{e}^{\boldsymbol{kt}}$, where $\boldsymbol{A}_0$ is the original amount at time $\boldsymbol{t} = 0$ and $\boldsymbol{k}$ is a constant of growth or decay (growth if $\boldsymbol{k} > 0$, decay if $\boldsymbol{k} < 0$.)

Example: The number $\boldsymbol{N}$ of bacteria present in a culture at time $\boldsymbol{t}$ hours obeys the law of uninhibited growth where $\boldsymbol{N}(\boldsymbol{t}) = \underset{N_0}{\underset{\uparrow}{1000}}\boldsymbol{e}^{0.01\boldsymbol{t}}$.

a) Determine the number of bacteria at $\boldsymbol{t} = 0$ hours.

$\boxed{1000}$

$$N(0) = 1000e^{(0.01)(0)} = 1000e^0 = 1000(1) = 1000$$

b) What is the growth rate of the bacteria?

$k = 0.01$ $\boxed{1\% \text{ per hour}}$

c) What will the population be after 4 hours?

$$N(4) = 1000e^{(0.01)(4)} \approx \boxed{1040.8}$$

d) When will the number of bacteria reach 1700?

$$1700 = 1000e^{0.01t}$$
$$1.7 = e^{0.01t}$$
$$\ln 1.7 = 0.01t$$
$$t = \frac{\ln 1.7}{0.01} \approx \boxed{53.1 \text{ hrs}}$$

e) When will the number of bacteria double?

$$2000 = 1000e^{0.01t}$$
$$2 = e^{0.01t}$$
$$\ln 2 = 0.01t$$
$$t = \frac{\ln 2}{0.01} \approx \boxed{69.3 \text{ hrs}}$$

Example: The annual growth rate of the world's population in 2005 was $\boldsymbol{k} = 1.15\% = 0.0115.$ The population of the world in 2005 was 6,451,058,790 people. Letting $\boldsymbol{t} = 0$ represent the year 2005, use the uninhibited growth model to predict the world's population in the year 2015. $\leftarrow t = 10$

$$A = A_0e^{kt}$$
$$A = 6{,}451{,}058{,}790\,e^{(0.0115)(10)}$$
$$\approx \boxed{7{,}237{,}271{,}501}$$

Example: Iodine 131 is a radioactive material that decays according to the function $A(t) = A_0 e^{-0.087t}$, where A_0 is the initial amount present and A is the amount present at time t (in days). Assume that a scientist has a sample of 100 grams of iodine 131.

A_0

a) What is the decay rate of iodine 131?

$k = -0.087$ $\boxed{8.7\% \text{ per day}}$

b) How much iodine 131 is left after 9 days?

$A(9) = 100e^{(-0.087)(9)} \approx \boxed{45.7\text{ g}}$

c) When will 70 grams of iodine 131 be left?

$70 = 100e^{-0.087t}$

$0.7 = e^{-0.087t}$

$\ln 0.7 = -0.087t$

$t = \frac{\ln 0.7}{-0.087} \approx \boxed{4.10 \text{ days}}$

d) What is the half-life of iodine 131? (when $A = \frac{1}{2}A_0$.)

$\frac{0.5\cancel{A_0}}{\cancel{A_0}} = \frac{\cancel{A_0}e^{-0.087t}}{\cancel{A_0}}$

$0.5 = e^{-0.087t}$

$\ln 0.5 = -0.087t$

$t = \frac{\ln 0.5}{-0.087} \approx \boxed{7.97 \text{ days}}$

Example: A piece of charcoal contains 30% of the carbon 14 that it originally had. When did the tree die from which the charcoal came? Use 5600 years as the half-life of carbon 14.

Step 1: Find k using half-life

$A = 0.5A_0$, $t = 5600$ yrs

$A = A_0e^{kt}$

$\frac{0.5\cancel{A_0}}{\cancel{A_0}} = \frac{\cancel{A_0}e^{k(5600)}}{\cancel{A_0}}$

$0.5 = e^{5600k}$

$\ln 0.5 = 5600k$

$k = \frac{\ln 0.5}{5600}$

$k \approx -1.238 \times 10^{-4}$

or -0.0001238

Step 2: Answer question

$A = 0.3A_0$ $t = ?$

$A = A_0e^{-0.0001238t}$

$\frac{0.3\cancel{A_0}}{\cancel{A_0}} = \frac{\cancel{A_0}e^{-0.0001238t}}{\cancel{A_0}}$

$0.3 = e^{-0.0001238t}$

$\ln 0.3 = -0.0001238t$

$t = \ln(0.3)/-0.0001238 \approx \boxed{9727 \text{ yrs}}$

Example: At 45°C, dinitrogen pentoxide decomposes into nitrous dioxide and oxygen according to the law of uninhibited decay. An initial amount of 0.25 M of dinitrogen pentoxide decomposes to 0.15 M in 17 minutes. How much dinitrogen pentoxide will remain after 30 minutes? How long will it take until 0.01 M of dinitrogen pentoxide remains?

Use to find k.

Step 1: Find k

$A_0 = 0.25M$, $A = 0.15M$, $t = 17$ min

$A = A_0e^{kt}$

$\frac{0.15}{0.25} = \frac{0.25e^{k(17)}}{0.25}$

$0.6 = e^{17k}$

$\ln 0.6 = 17k$

$k = \frac{\ln 0.6}{17} \approx -0.030$

$A(t) = 0.25e^{-0.030}$

Question 1: Amt after 30 minutes

$A(30) = 0.25e^{-0.030(30)}$

$\approx \boxed{0.101\text{ M}}$

Question 2: How long until $A = 0.01M$?

$\frac{0.01}{0.25} = \frac{0.25e^{-0.030t}}{0.25}$

$0.04 = e^{-0.030t}$

$\ln 0.04 = -0.030t$

$t = \frac{\ln 0.04}{-0.030} \approx \boxed{107.1 \text{ min}}$