Logarithmic and Exponential Equations

Solving Logarithmic Equations

- To avoid extraneous solutions, determine the domain of the variable first.
- Use the properties of logarithms to simplify the equation as much as possible.
- If the equation looks like log = #:
  1. Get the log by itself.
  2. Rewrite as an exponential function: $y = \log_a x \Leftrightarrow x = a^y$
- If the equation looks like log = log:
  1. Isolate one log on each side of the equation.
  2. Use the property $\log_a M = \log_a N \Leftrightarrow M = N$.
- Check the solutions to make sure they are part of the domain. Any solutions that aren't part of the domain are extraneous – they aren't actually solutions!

Examples:

a) $\log_3(3x-1) = 2$

$3x-1>0$
$x>\frac{1}{3}$

$3^2 = 3x-1$
$9 = 3x-1$
$10 = 3x$
$\boxed{x = \frac{10}{3}}$

b) $3\log_2(x-1) + \log_2 4 = 5$

$x-1>0$
$x>1$

$\log_2(x-1)^3 + 2 = 5$
$\log_2(x-1)^3 = 3$
$\sqrt[3]{2^3} = \sqrt[3]{(x-1)^3}$
$2 = x-1$
$\boxed{x = 3}$

c) $-2\log_4 x = \log_4 9$

$x>0$

$\log_4 x^{-2} = \log_4 9$
$x^{-2} = 9$
$\frac{1}{x^2} = 9$
$1 = 9x^2$
$x^2 = \frac{1}{9}$
$x = \pm\frac{1}{3}$
but … $x>0$
so $\boxed{x = \frac{1}{3}}$

d) $\ln(x+1) - \ln x = 2$

$x+1>0$
$x>-1$
$x>0$

$\ln\left(\frac{x+1}{x}\right) = 2$
$e^2 = \frac{x+1}{x}$
$e^2x = x+1$
$e^2x - x = 1$
$x(e^2-1) = 1$
$\boxed{x = \frac{1}{e^2-1} \approx 0.157}$

e) $\log_6(x+4) + \log_6(x+3) = 1$

$x+4>0$
$x>-4$
$x+3>0$
$x>-3$

$\log_6[(x+4)(x+3)] = 1$
$6^1 = x^2+7x+12$
$6 = x^2+7x+12$
$x^2+7x+6 = 0$
$(x+6)(x+1) = 0$
$\cancel{x=-6}$ $\boxed{x=-1}$

f) $\log_a x + \log_a(x-2) = \log_a(x+4)$

$x>0$
$x-2>0$
$x>2$
$x+4>0$
$x>-4$

$\log_a[x(x-2)] = \log_a(x+4)$
$x(x-2) = x+4$
$x^2-2x = x+4$
$x^2-3x-4 = 0$
$(x-4)(x+1) = 0$
$\boxed{x=4}$ $\cancel{x=-1}$

Solving Exponential Equations

- If possible, make the bases the same, set exponents equal, and solve: $\boldsymbol{a}^{u} = \boldsymbol{a}^{v} \Leftrightarrow \boldsymbol{u} = \boldsymbol{v}$.
- If there is only one exponent:
  1. Isolate the base with the exponent.
  2. Rewrite as a logarithm: $\boldsymbol{x} = \boldsymbol{a}^{y} \Leftrightarrow \boldsymbol{y} = \log_{a} \boldsymbol{x}$
- If there are multiple exponents with variables:
  1. Taking the logarithm of both sides: $\boldsymbol{M} = \boldsymbol{N} \Leftrightarrow \log_{a} \boldsymbol{M} = \log_{a} \boldsymbol{N}$.
  2. Use the power property ($\log_{a} \boldsymbol{M}^{r} = \boldsymbol{r} \log_{a} \boldsymbol{M}$) to bring the variables out of the exponents.
  3. Get all the terms with the variable on one side and everything else on the other side.
  4. If necessary, factor out the variable, then solve.

Examples:

a) $3^{x+5} = 27^{x}$

$3^{x+5} = (3^{3})^{x}$

$3^{x+5} = 3^{3x}$

$x + 5 = 3x$

$5 = 2x$ $\quad \boxed{x = \frac{5}{2}}$

b) $10^{2x-7} = 3$

$\log_{10}$

$\log 3 = 2x - 7$

$\log(3) + 7 = 2x$

$\boxed{x = \frac{\log(3) + 7}{2} \approx 3.739}$

c) $e^{3x-2} = 7$

$\log_{e}$

$\ln 7 = 3x - 2$

$\ln(7) + 2 = 3x$

$\boxed{x = \frac{\ln(7) + 2}{3} \approx 1.315}$

d) $\frac{4e^{0.5x}}{4} = \frac{5}{4}$

$e^{0.5x} = 5/4$

$\ln(5/4) = 0.5x$

$\boxed{x = \frac{\ln(5/4)}{0.5} \approx 0.446}$

e) $2^{-x} = 1.5$

$\log_{2} 1.5 = -x$

$x = -\log_{2} 1.5 = \frac{-\log 1.5}{\log 2} \approx \boxed{-0.585}$

Change-of-base formula

f) $\frac{0.3(4^{0.2x})}{0.3} = \frac{0.2}{0.3}$

$4^{0.2x} = \frac{2}{3}$

$\log_{4}\left(\frac{2}{3}\right) = 0.2x$

change-of-base formula

$x = \frac{\log_{4}(2/3)}{0.2} = \frac{\log(2/3)/\log 4}{0.2} \approx \boxed{-1.462}$

g) $e^{x+3} = \pi^{x}$

use ln

$\ln e^{x+3} = \ln \pi^{x}$

What exponent do you put on e to get e^{x+3}?

$x + 3 = x \ln \pi$
$\quad -x \qquad\qquad -x$

$3 = x\ln(\pi) - x$

$3 = x[\ln(\pi) - 1]$

$\boxed{x = \frac{3}{\ln(\pi) - 1} \approx 20.728}$

h) $3^{x-4} = 7^{5x+1}$

$\log 3^{x-4} = \log 7^{5x+1}$

$(x-4)\log 3 = (5x+1)\log 7$

$x\log 3 - 4\log 3 = 5x\log 7 + \log 7$
$-5x\log 7 \quad +4\log 3 \qquad -5x\log 7 \quad +4\log 3$

$x\log 3 - 5x\log 7 = \log 7 + 4\log 3$

$x(\log 3 - 5\log 7) = \log 7 + 4\log 3$

$\boxed{x = \frac{\log 7 + 4\log 3}{\log 3 - 5\log 7} \approx -0.735}$

Example: The blood alcohol concentration (BAC) is the concentration of alcohol in a person's bloodstream. The relative risk of having an accident while driving a car is given by the equation $\boldsymbol{R} = \boldsymbol{e}^{kx}$, where $\boldsymbol{R}$ is the relative risk (how many times more likely a person with a certain BAC is to have a car accident than a person who has not been drinking), $\boldsymbol{x}$ is the BAC (expressed as a percentage), and $\boldsymbol{k}$ is a constant.

a) If the relative risk is 1.4 when the blood concentration is 0.02%, find $\boldsymbol{k}$.

$$R = e^{kx}$$

$$1.4 = e^{k(0.02)}$$

$$\ln 1.4 = 0.02k$$

$$k = \frac{\ln 1.4}{0.02} \approx \boxed{16.824}$$

$$R = e^{16.824x}$$

b) Using $\boldsymbol{k}$ from part a), find the relative risk if the blood alcohol concentration is 0.17%.

$$R = e^{(16.824)(0.17)} \approx \boxed{17.5}$$

17.5x more likely to have accident than someone who hasn't been drinking

c) What BAC corresponds to a relative risk of 100?

$$100 = e^{16.824x}$$

$$\ln 100 = 16.824x$$

$$x = \frac{\ln 100}{16.824} \approx \boxed{0.27\%}$$