## Properties of Logarithms

### A *Very* Brief History of Logarithms:

In the late 1500's, a Scottish mathematician named John Napier was searching for a way to simplify calculations. Scientists often needed to multiply together long lists of numbers, and doing this by hand took a very long time. Napier was looking for a method that would change a multiplication problem into an addition problem, which was easier to do by hand. He succeeded in publishing a table of values in 1614 that he called logarithms. A simplified example of how his table worked is found below:

| $x$ | 16 | 32 | 64 | 128 | 256 | 512 | 1024 | 2048 | 4096 |
|---|---|---|---|---|---|---|---|---|---|
| $\log_2 x$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |

**Example:** Calculate $256 \times 16$.

1. Look up the logarithms of each number being multiplied: $\log_2 256 = 8$ and $\log_2 16 = 4$.

2. Add the two logarithms together: $8 + 4 = 12$.

3. Look up the number that has a logarithm of 12, which is 4096. Therefore, $256 \times 16 = 4096$.

This works because when you multiply two numbers with a common base, you add their exponents (remember that logarithms are exponents): $256 \times 16 = 2^8 \cdot 2^4 = 2^{8+4} = 2^{12}$.

We were able to simplify a multiplication problem to a simple addition problem. Napier's logarithms were a bit different from those we use today (and more complicated), but they followed the same principal. Napier worked with the English mathematician Henry Briggs to perfect the system. In 1624, Briggs published tables giving the base-10 logarithms of the natural numbers from 1 to 20,000 and 90,000 to 100,000, each to 14 decimal places. In 1628, the Dutch mathematician Adrian Vlacq published a complete table from 1 to 100,000. Finding these values by hand required a ridiculous amount of work (to find $\log 2$, Briggs had to carry out 47 consecutive square roots by hand), but once the work was done, it made life incredibly easier for scientists and mathematicians. Here is a table of logs from 1.00 to 1.49:

### TABLE OF COMMON LOGARITHMS (BASE 10)

|  | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
|---|---|---|---|---|---|---|---|---|---|---|
| **1.0** | 0.000000 | 0.004321 | 0.008600 | 0.012837 | 0.017033 | 0.021189 | 0.025306 | 0.029384 | 0.033424 | 0.037426 |
| **1.1** | 0.041393 | 0.045323 | 0.049218 | 0.053078 | 0.056905 | 0.060698 | 0.064458 | 0.068186 | 0.071882 | 0.075547 |
| **1.2** | 0.079181 | 0.082785 | 0.086360 | 0.089905 | 0.093422 | 0.096910 | 0.100371 | 0.103804 | 0.107210 | 0.110590 |
| **1.3** | 0.113943 | 0.117271 | 0.120574 | 0.123852 | 0.127105 | 0.130334 | 0.133539 | 0.136721 | 0.139879 | 0.143015 |
| **1.4** | 0.146128 | 0.149219 | 0.152288 | 0.155336 | 0.158362 | 0.161368 | 0.164353 | 0.167317 | 0.170262 | 0.173186 |

**Memorize the following:**

*Definition of Logarithm:* $y = \log_a x \iff a^y = x$

*Properties of Logarithms:* For any positive numbers $M$, $N$, and $a$, where $a \neq 1$ and $r$ is any real number:

$$\log_a 1 = 0 \qquad\qquad \log_a a = 1 \qquad\qquad a^{\log_a M} = M \qquad\qquad \log_a a^r = r$$

$$\log_a (MN) = \log_a M + \log_a N \qquad\qquad\qquad \log_a M^r = r\log_a M$$

$$\log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N \qquad\qquad\qquad \log_a M = \log_a N \iff M = N$$

*Change of Base Formula:* $\log_a M = \dfrac{\log_b M}{\log_b a} \qquad\qquad \log_a M = \dfrac{\log M}{\log a} \qquad\qquad \log_a M = \dfrac{\ln M}{\ln a}$

**Examples:** Find the exact value of each expression. (Do not use a calculator).

a) $\log_{0.6} 0.6^{-3.2}$ 
b) $8^{\log_8 3}$ 
c) $\log 10^{-5}$ 
d) $e^{\ln 12}$

**Examples:** Use the change of base formula to evaluate each logarithm.

a) $\log_6 9$ 
b) $\log_{\sqrt{2}} 7$ 
c) $\log_\pi \sqrt{3}$

**Examples:** Use properties of logarithms to find the exact value of each expression. (Do not use a calculator).

a) $\log_6 18 - \log_6 3$ 
b) $5^{\log_5 6 + \log_5 7}$ 
c) $\log_3 8 \cdot \log_8 9$

**Examples:** Write each expression as a sum/difference of logarithms. Express powers as factors.

a) $\log_7\left(2x^5\right)$ 
b) $\log_7\left(2x\right)^5$ 
c) $\ln\left(xe^x\right)$

c) $\log_2\left(\dfrac{x^2 y^3}{8\sqrt{z}}\right)$

d) $\log\sqrt[3]{mn}$

e) $\ln\left[\dfrac{(x-4)^2}{x^2-1}\right]^{2/3}$

**Examples:** Write each expression as a single logarithm.

a) $3\log_5 v + 4\log_5 w$

b) $\log_4\left(x^2-1\right) - 5\log_4\left(x+1\right)$

c) $\dfrac{\log_2 k}{2} - 5\log_2 m - 3\log_2 n$

d) $\dfrac{1}{4}\left(\log_a x - \log_a y\right)$

e) $\log\left(\dfrac{x^2-2x-3}{x^2-4}\right) - \log\left(\dfrac{x^2+7x+6}{x+2}\right)$