Logarithmic Functions

Question: What is the inverse of an exponential function? How do you solve for a variable that is in an exponent?

Find the inverse of $f(x) = 2^x$.

1. Write $f(x)$ as y. $y = 2^x$
2. Swap x and y. $x = 2^y$
3. Solve for y. y = the exponent to which we raise 2 to get x.
4. Rename y as $f^{-1}(x)$ $f^{-1}(x)$ = the exponent to which we raise 2 to get x.

We need a new symbol to replace the words: "The exponent to which we raise 2 to get x"

Symbol: $\log_2 x$ **Pronounced: "the logarithm, base 2, of *x*" or "log, base 2, of *x*"**

★LOGARITHMS ARE EXPONENTS!★

Logarithm: $\log_b a$ means the ***exponent*** to which we raise b to get a.

- b is called the ***base*** of the logarithm (the number being raised to the exponent).
- a is called the ***argument*** of the logarithm (the number you get when you raise the base to the exponent).

Converting Between Logarithmic and Exponential Form:

If b is a positive number other than 1, and a is a positive number:

$$\log_b a = x \quad \Leftrightarrow \quad b^x = a$$

$$(\text{"}\log_{\text{base}} \text{argument} = \text{exponent"} \quad \Leftrightarrow \quad \text{"base}^{\text{exponent}} = \text{argument"})$$

Common Logarithms and Natural Logarithms

- Logarithms with base 10 are called "common logarithms".
  - $\log_{10} x$ is written as $\log x$.
- Logarithms with base e are called "natural logarithms".
  - $\log_e x$ is written as $\ln x$.

Example: Change each exponential expression to an equivalent expression involving a logarithm.

a) $5^4 = 625$ b) $n^3 = 64$ c) $3^2 = w$ d) $e^6 = k$ e) $10^y = 73$

Example: Change each logarithmic expression to an equivalent expression involving an exponent.

a) $\log_3 81 = 4$ b) $\log_m 25 = 2$ c) $\log_p q = r$ d) $\ln 5 = x$ e) $\log x = 3$

Evaluating Logarithms: It is helpful to replace "log" with the word "power".

- Instead of "$\log_2 8$," think "$\text{power}_2\, 8$." Ask yourself, what power of 2 equals 8?
  - The answer would be 3 because $2^3 = 8$.

Example: Find the exact value of

a) $\log_3 9$ b) $\log_{1/2}(1/32)$ c) $\log_6 1$ d) $\log 0.0001$ e) $\log_7 \sqrt{7}$ f) $\ln \sqrt[5]{e^3}$

Domain of a Logarithmic Function

The logarithmic function $y = \log_a x$ is the inverse of the exponential function $y = a^x$.

Domain of a logarithmic function = Range of the exponential function that is its inverse

Range of a logarithmic function = Domain of the exponential function that is its inverse $= (-\infty, \infty)$

$y = \log_a x$ (defining equation: $x = a^y$)

Domain: $(0, \infty)$ Range: all real numbers

★ **You can't take the log of zero or of a negative** because it is impossible to get zero or a negative by raising a positive base to an exponent.

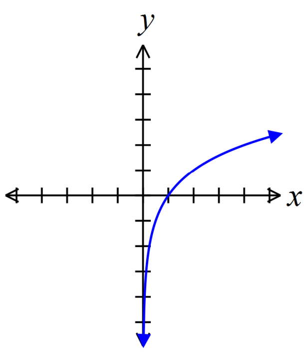
★ **The argument of a logarithmic function must be greater than zero.**

Example: Find the domain of each logarithmic function

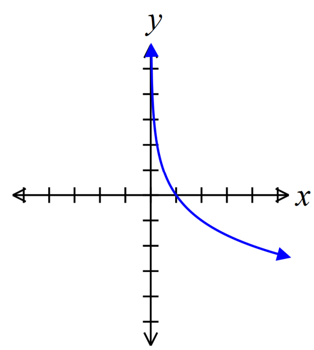
a) $f(x) = \log_2(x+3)$ b) $g(x) = \log_5(10-2x)$ c) $h(x) = \log_{\frac{1}{2}}|x|$

Graphs of Logarithmic Functions

$f(x) = \log_a x,\ a > 1$



$f(x) = \log_a x,\ 0 < a < 1$



Properties of the Logarithmic Function $f(x) = \log_a x$

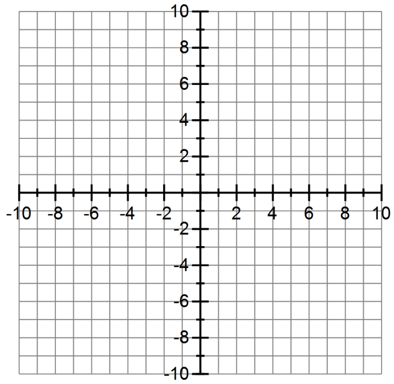
1. The domain is the set of all positive real numbers; the range is the set of all real numbers.
2. The x-intercept is 1. There is no y-intercept.
3. The y-axis $(x = 0)$ is a vertical asymptote of the graph.
4. The logarithmic function is decreasing if $0 < a < 1$ and increasing if $a > 1$. The function is one-to-one.
5. The graph of f contains the points $(1,0)$, $(a,1)$, and $\left(\frac{1}{a}, -1\right)$.
6. The graph of f is smooth and continuous, with no corners, gaps, or cusps.

Graphing Logarithmic Functions:

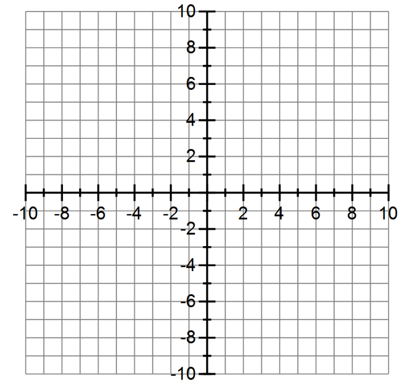
1. Solve the equation for x by rewriting it as an exponential function. $y = \log_a x \Leftrightarrow a^y = x$
   - ★ When you do this, get the logarithm by itself on one side of the equation first, then rewrite.
2. Choose y-values, and plug them in to find the x-values.
   - ★ Choose y-values that will make the *exponents* be –2, –1, 0, 1, and 2. If the exponent in the equation is $y+3$, choose –5, –4, –3, –2, and –1, because when you add 3 to these y's, you will get –2, –1, 0, 1, and 2. If the exponent in the equation is $y/3$, choose –6, –3, 0, 3, and 6, because you will divide these y's by 3 to get the exponents.
3. Plot your points and connect them to form a smooth curve.

Examples: Graph the following functions. State the domain and range, and label any asymptotes.

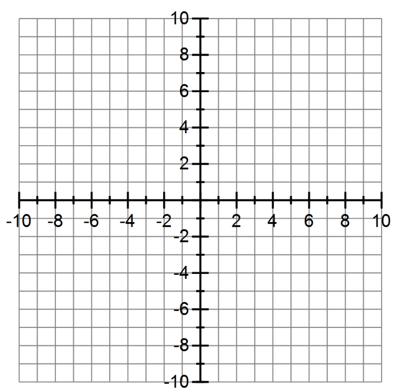
a) $y = \log_2 x$



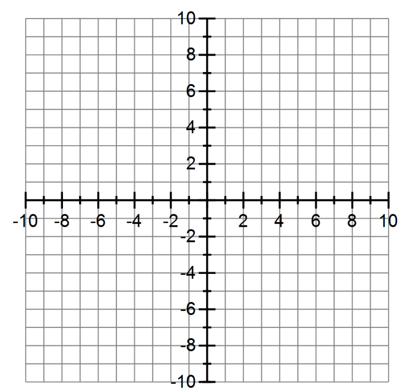
b) $y = -\log_{1/3} x$



c) $f(x) = \log_3 (x-1)$

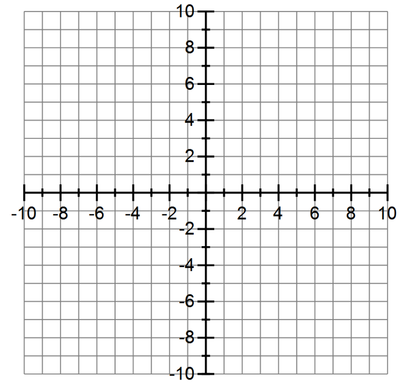


d) $f(x) = \log_{1/2} x + 2$



Example: $f(x) = 2\ln(x-3)$

a) Find the domain of the logarithmic function.
b) Graph $f(x)$.
c) Find the range and vertical asymptote of f.
d) Find f^{-1}, the inverse of f.
e) Graph f^{-1}.



Example: $f(x) = -\log(x+4)$

a) Find the domain of the logarithmic function.
b) Graph $f(x)$.
c) Find the range and vertical asymptote of f.
d) Find f^{-1}, the inverse of f.
e) Graph f^{-1}.