Logarithmic Functions

Question: What is the inverse of an exponential function? How do you solve for a variable that is in an exponent?

Find the inverse of $f(x) = 2^x$.

1. Replace $f(x)$ with *y*. $y = 2^x$
2. Interchange *x* and *y*. $x = 2^y$
3. Solve for *y*. $y =$ the exponent to which we raise 2 to get *x*.
4. Replace *y* with $f^{-1}(x)$ $f^{-1}(x) =$ the exponent to which we raise 2 to get *x*.

We need a new symbol to replace the words: "The exponent to which we raise 2 to get *x*":

$\log_2 x$ means "the exponent to which we raise 2 to get *x*."

Pronounced "the logarithm, base 2, of *x*" or "log, base 2, of *x*"

★LOGARITHMS ARE EXPONENTS!★

Logarithm: $\log_b a$ means the ***exponent*** to which we raise b to get a.

- b is called the ***base*** of the logarithm (the number being raised to the exponent).
- a is called the ***argument*** of the logarithm (the number you get when you raise the base to the exponent).

The ***logarithmic function of base a***, where $a > 0$ and $a \neq 1$ is denoted by $y = \log_a x$ and is defined by

$$y = \log_a x \text{ if and only if } x = a^y.$$

Example: Change each exponential expression to an equivalent expression involving a logarithm.

a) $5^x = 625$

$\log_5 625 = x$

b) $x^3 = 64$

$\log_x 64 = 3$

c) $3^2 = x$

$\log_3 x = 2$

Example: Change each logarithmic expression to an equivalent expression involving an exponent.

a) $\log_3 x = 5$

$3^5 = x$

b) $\log_e 5 = x$

$e^x = 5$

c) $\log_m 2 = n$

$m^n = 2$

Evaluating Logarithms: It is helpful to replace "log" with the word "power".

- Instead of "$\log_2 8$," think "$\text{power}_2\, 8$." Ask yourself, what power of 2 equals 8?
  - The answer would be 3 because $2^3 = 8$.

Example: Find the exact value of

a) $\log_3 9 = \boxed{2}$

$3^? = 9$

b) $\log_2 32 = \boxed{5}$

$2^? = 32$

c) $\log_6 1 = \boxed{0}$

$6^? = 1$

d) $\log_5 \frac{1}{125} = \boxed{-3}$

$5^? = \frac{1}{125}$

e) $\log_7 \sqrt{7} = \boxed{\frac{1}{2}}$

$7^? = \sqrt{7}$

$7^{1/2} = \sqrt{7}$

Domain of a Logarithmic Function

The logarithmic function $y = \log_a x$ is the inverse of the exponential function $y = a^x$.

Domain of the logarithmic function = Range of the exponential function

Range of the logarithmic function = Domain of the exponential function $= (-\infty, \infty)$

Inverse: $y = a^x$

$y = \log_a x$ (defining equation: $x = a^y$)

Domain: $(0, \infty)$ Range: all real numbers

★ **You can't take the log of zero or of a negative** because it is impossible to get zero or a negative by raising a positive base to an exponent.

★ **The argument of a logarithmic function must be greater than zero.**

Example: Find the domain of each logarithmic function

a) $f(x) = \log_2(x+3)$

$x + 3 > 0$

$\boxed{\{x \mid x > -3\} \quad (-3, \infty)}$

b) $g(x) = \log_5(10 - 2x)$

$10 - 2x > 0$

$10 > 2x$

$5 > x$

$\boxed{\{x \mid x < 5\} \quad (-\infty, 5)}$

or $\dfrac{-2x}{-2} > \dfrac{-10}{-2}$ Flip sign!

$x < 5$

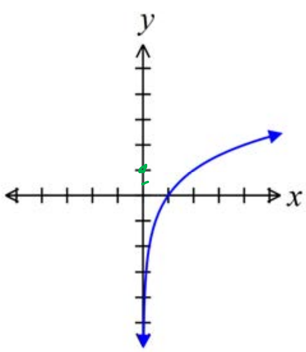
c) $h(x) = \log_{\frac{1}{2}}|x|$

$|x| > 0$ ← True for everything but 0

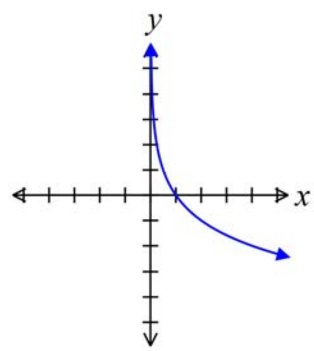
$\boxed{\{x \mid x \neq 0\} \quad (-\infty, 0) \cup (0, \infty)}$

Graphs of Logarithmic Functions

$f(x) = \log_a x,\ a > 1$



$f(x) = \log_a x,\ 0 < a < 1$



Properties of the Logarithmic Function $f(x) = \log_a x$

1. The domain is the set of all positive real numbers; the range is the set of all real numbers.
2. The x-intercept is 1. There is no y-intercept.
3. The y-axis $(x = 0)$ is a vertical asymptote of the graph.
4. The logarithmic function is decreasing if $0 < a < 1$ and increasing if $a > 1$. The function is one-to-one.
5. The graph of f contains the points $(1, 0)$, $(a, 1)$, and $\left(\frac{1}{a}, -1\right)$.
6. The graph of f is smooth and continuous, with no corners, gaps, or cusps.

★ **Note:** It is often easier to graph a logarithmic function if you rewrite it as an exponential function first.

Graphing Logarithmic Functions:

1. Solve the equation for x by rewriting it as an exponential function. $y = \log_a x \Leftrightarrow a^y = x$
   - ★ When you do this, get the logarithm by itself on one side of the equation first, then rewrite.
2. Choose y-values, and plug them in to find the x-values.
3. Plot your points and connect them to form a smooth curve.

$x>0$

$x>0$

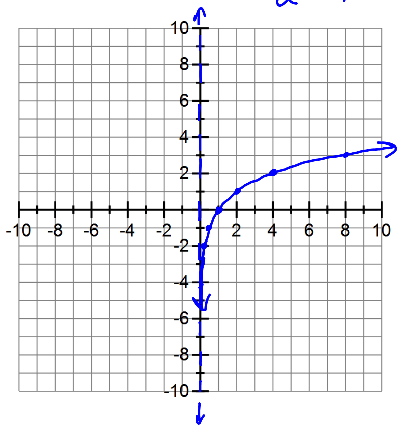
Examples: Graph the following functions. State the domain and range, and label any asymptotes.

a) $y=\log_2 x$

$2^y = x$

pick these

x	y
$2^{-2}=\frac{1}{4}$	-2
$2^{-1}=\frac{1}{2}$	-1
$2^0=1$	0
$2^1=2$	1
$2^2=4$	2



$x=0$

Domain: $(0,\infty)$

Range: $(-\infty,\infty)$

b) $y=-\log_{1/3} x$

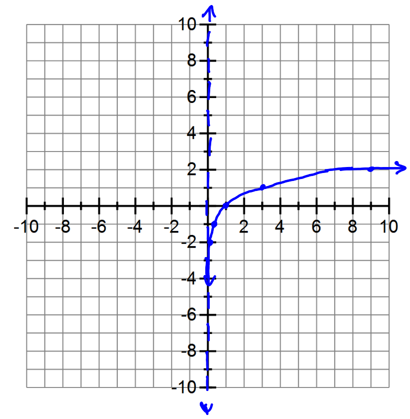
Get log by itself:

$-y=\log_{1/3} x$

Rewrite:

$\left(\frac{1}{3}\right)^{-y}=x$

x	y
$\left(\frac{1}{3}\right)^{-(-2)}=\frac{1}{9}$	-2
$\left(\frac{1}{3}\right)^{-(-1)}=\frac{1}{3}$	-1
$\left(\frac{1}{3}\right)^{-0}=1$	0
$\left(\frac{1}{3}\right)^{-1}=3$	1
$\left(\frac{1}{3}\right)^{-2}=9$	2



$x=0$

Domain: $(0,\infty)$

Range: $(-\infty,\infty)$

c) $f(x)=\log_3(x-1)$

Domain: $x-1>0$

$\{x \mid x>1\}$

V.A.: $x=1$

D: $(1,\infty)$

R: $(-\infty,\infty)$

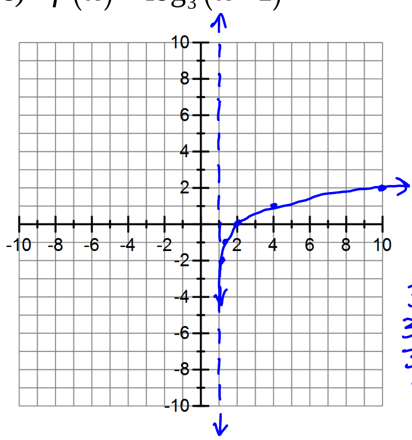
$y=\log_3(x-1)$

Rewrite:

$3^y = x-1$

$x = 3^y+1$

x	y
$3^{-2}+1=1\frac{1}{9}$	-2
$3^{-1}+1=1\frac{1}{3}$	-1
$3^0+1=2$	0
$3^1+1=4$	1
$3^2+1=10$	2



$x=1$

d) $f(x)=\log_{1/2} x+2$

$x>0$

$-2 \qquad -2$

Get log by itself:

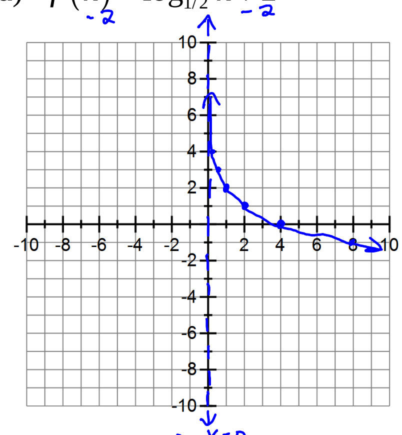
$y-2=\log_{1/2} x$

Rewrite:

$\left(\frac{1}{2}\right)^{y-2}=x$

pick y's 2 higher than normal because I will subtract 2 from y before using answer as exponent

x	y
$\left(\frac{1}{2}\right)^{0-2}=4$	0
$\left(\frac{1}{2}\right)^{1-2}=2$	1
$\left(\frac{1}{2}\right)^{2-2}=1$	2
$\left(\frac{1}{2}\right)^{3-2}=\frac{1}{2}$	3
$\left(\frac{1}{2}\right)^{4-2}=\frac{1}{4}$	4



$x=0$

D: $(0,\infty)$

R: $(-\infty,\infty)$

Natural Logarithms: If the base of a logarithmic function is the number *e*, then we have the natural logarithm function (abbreviated ln). That is, $y = \ln x$ means $y = \log_e x$. $y = \ln x$ if and only if $x = e^y$.

Example: $f(x) = -\ln(x+3)$

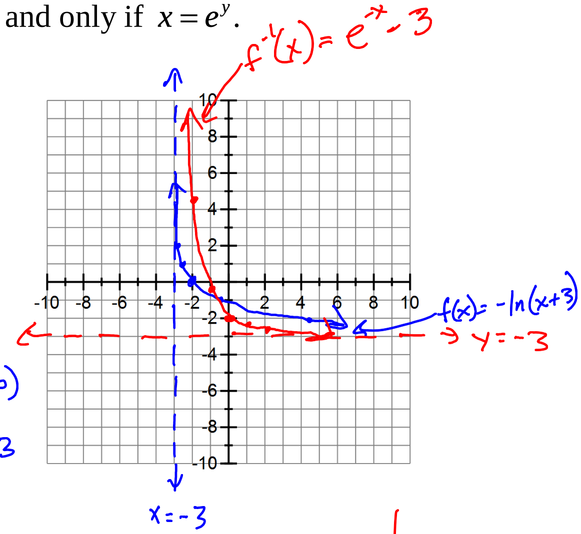
a) Find the domain of the logarithmic function.

b) Graph $f(x)$.

c) Find the range and vertical asymptote of f.

d) Find f^{-1}, the inverse of f.

e) Graph f^{-1}.



a) Domain: $x + 3 > 0$

$x > -3$ $\quad (-3, \infty)$

c) Range: $(-\infty, \infty)$

V. asymp: $x = -3$

b) $y = -\ln(x+3)$

Get log by itself: $-y = \ln(x+3)$

Rewrite: $e^{-y} = x + 3$

$x = e^{-y} - 3$

x	y
$e^{-(-2)} - 3 \approx 4.4$	-2
$e^{-(-1)} - 3 \approx -.3$	-1
$e^{-0} - 3 = -2$	0
$e^{-1} - 3 \approx -2.6$	1
$e^{-2} - 3 \approx -2.9$	2

d) Inverse: $f^{-1}(x) = e^{-x} - 3$

Swap x & y

e)

x	y
-2	4.4
-1	$-.3$
0	-2
1	-2.6
2	-2.9

Common Logarithmic Function: If the base of a logarithmic function is the number 10, then we have the common logarithmic function. If the base of the logarithmic function is not indicated, it is understood to be 10. That is, $y = \log x$ means $y = \log_{10} x$. $y = \log x$ if and only if $x = 10^y$.

Example: $f(x) = 2\log(x-3)$

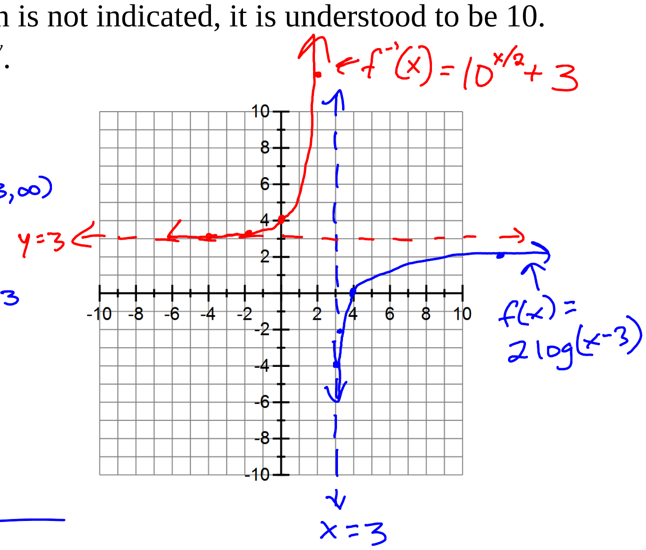
a) Find the domain of the logarithmic function. $\quad x - 3 > 0$, $x > 3$ $\quad (3, \infty)$

b) Graph $f(x)$.

c) Find the range and vertical asymptote of f. $\quad$ R: $(-\infty, \infty)$ $\quad$ V.a. $x = 3$

d) Find f^{-1}, the inverse of f.

e) Graph f^{-1}.



$y = 2\log(x-3)$

Get log by itself: $\frac{y}{2} = \log(x-3)$

Rewrite: $10^{y/2} = x - 3$

$x = 10^{y/2} + 3$

plug in even y's because I need to divide them by 2

x	y
$10^{-4/2} + 3 = 3\frac{1}{100}$	-4
$10^{-2/2} + 3 = 3\frac{1}{10}$	-2
$10^{\frac{0}{2}} + 3 = 4$	0
$10^{\frac{2}{2}} + 3 = 13$	2
$10^{\frac{4}{2}} + 3 = 103$	4

d) $f^{-1}(x) = 10^{x/2} + 3$

e)

x	y
-4	$3\,^1/_{100}$
-2	$3\,^1/_{10}$
0	4
2	13
4	103