Exponential Functions

Laws of Exponents**:** If *m*, *n*, *a*, and *b* are real numbers with $a > 0$ and $b > 0$, then

exponent

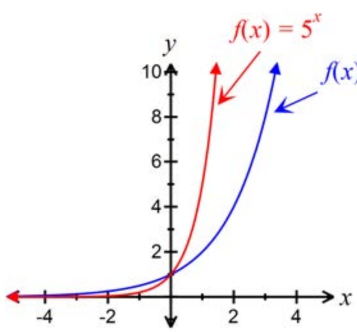
$$a^m \cdot a^n = a^{m+n} \qquad \frac{a^m}{a^n} = a^{m-n} \qquad \left(a^m\right)^n = a^{mn} \qquad (ab)^m = a^m b^m \qquad a^{-m} = \frac{1}{a^m} = \left(\frac{1}{a}\right)^m \qquad a^0 = 1$$

base

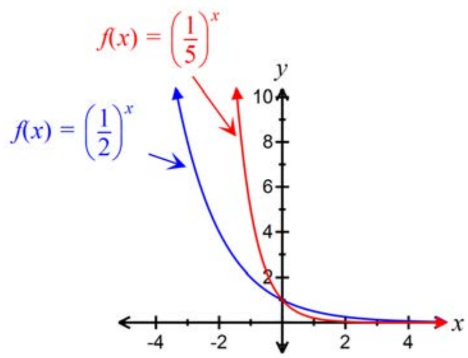
An ***exponential function*** is a function of the form $f(x) = a^x$, where a is a positive real number $(a > 0)$ and $a \neq 1$. The domain of f is the set of all real numbers.

Properties of the Exponential Function $f(x) = a^x$, $a > 0$, $a \neq 1$

- Domain: $(-\infty, \infty)$; Range: $(0, \infty)$
- There are no *x*-intercepts; the *y*-intercept is 1.
- The *x*-axis $(y = 0)$ is a horizontal asymptote.
    - For $a > 1$, the graph approaches the *x*-axis as $x \to -\infty$.
    - For $0 < a < 1$, the graph approaches the *x*-axis as $x \to \infty$.
- $f(x) = a^x$ is one-to-one.
    - For $a > 1$, $f(x) = a^x$ is an increasing function.
    - For $0 < a < 1$, $f(x) = a^x$ is a decreasing function.
- The graph of f contains the points $\left(-1, \frac{1}{a}\right)$, $(0, 1)$, and $(1, a)$.
- The graph of f is smooth and continuous, with no corners, gaps, or cusps.



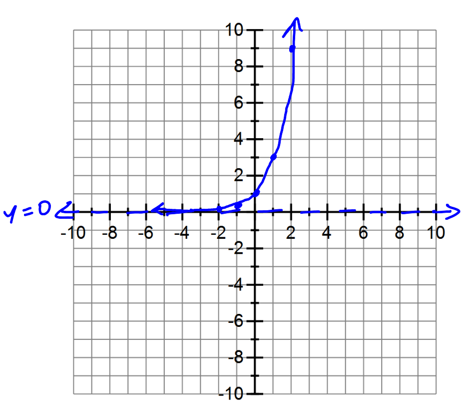
$a > 1$



$0 < a < 1$

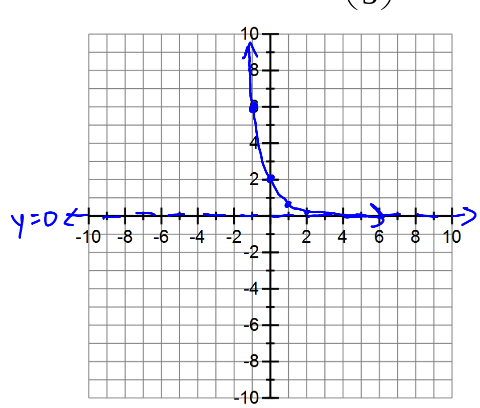
Examples:

a) Graph $f(x) = 3^x$.



x	y
-2	$3^{-2} = \frac{1}{9}$
-1	$3^{-1} = \frac{1}{3}$
0	$3^0 = 1$
1	$3^1 = 3$
2	$3^2 = 9$

b) Graph $f(x) = \underset{\text{vertical stretch}}{2} \cdot \left(\frac{1}{3}\right)^x$.   Parent: $y = \left(\frac{1}{3}\right)^x$

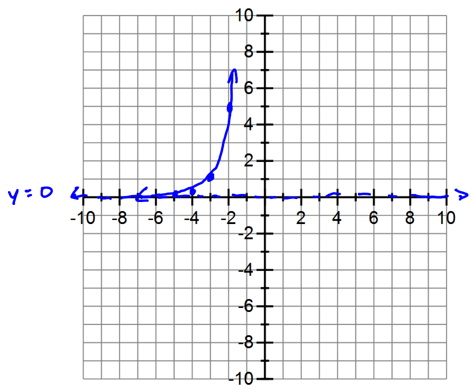


x	y
-2	$2\left(\frac{1}{3}\right)^{-2} = 2(3)^2 = 18$
-1	$2\left(\frac{1}{3}\right)^{-1} = 2(3)^1 = 6$
0	$2\left(\frac{1}{3}\right)^0 = 2(1) = 2$
1	$2\left(\frac{1}{3}\right)^1 = \frac{2}{3}$
2	$2\left(\frac{1}{3}\right)^2 = 2\left(\frac{1}{9}\right) = \frac{2}{9}$

c) Graph $f(x)=5^{x+3}$.

parent: $y=5^x$

$\leftarrow 3$



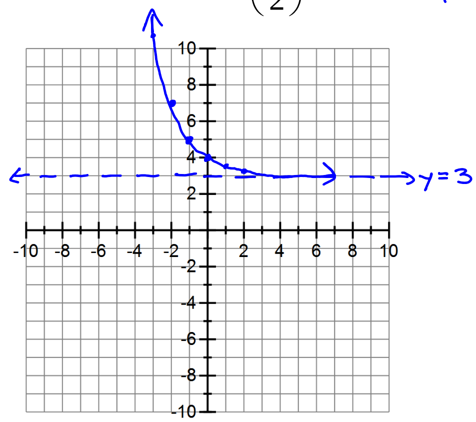
pick x's 3 to the left of usual

x	y
-5	$5^{-5+3}=5^{-2}=\frac{1}{25}$
-4	$5^{-4+3}=5^{-1}=\frac{1}{5}$
-3	$5^{-3+3}=5^{0}=1$
-2	$5^{-2+3}=5^{1}=5$
-1	$5^{-1+3}=5^{2}=25$

d) Graph $f(x)=\left(\frac{1}{2}\right)^x+3$.

parent: $y=\left(\frac{1}{2}\right)^x$

up 3

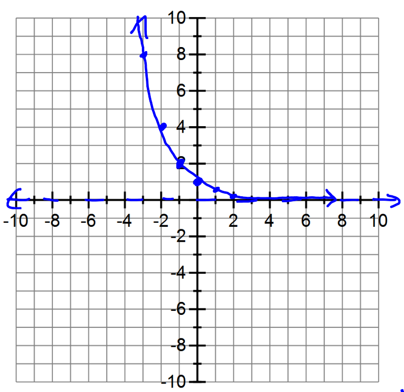


x	y
-2	$\left(\frac{1}{2}\right)^{-2}+3=7$
-1	$\left(\frac{1}{2}\right)^{-1}+3=5$
0	$\left(\frac{1}{2}\right)^{0}+3=4$
1	$\left(\frac{1}{2}\right)^{1}+3=3\frac{1}{2}$
2	$\left(\frac{1}{2}\right)^{2}+3=3\frac{1}{4}$

e) Graph $f(x)=2^{-x}$.

parent: $y=2^x$

reflect over y-axis



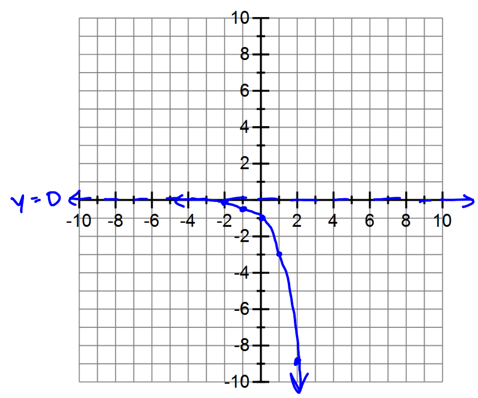
x	y
-2	$2^{-(-2)}=4$
-1	$2^{-(-1)}=2$
0	$2^{0}=1$
1	$2^{-1}=\frac{1}{2}$
2	$2^{-2}=\frac{1}{4}$

same as $y=\left(\frac{1}{2}\right)^x$

f) Graph $f(x)=-3^x$

parent: $y=3^x$

reflected over x-axis

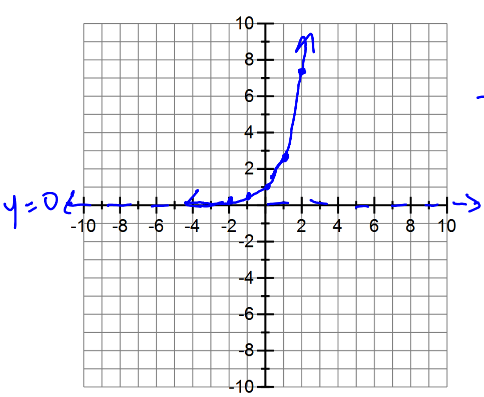


x	y
-2	$-3^{-2}=-\frac{1}{9}$
-1	$-3^{-1}=-\frac{1}{3}$
0	$-3^{0}=-1$
1	$-3^{1}=-3$
2	$-3^{2}=-9$

The **number *e*** (approximately 2.71828…) is defined as the number that the expression $\left(1+\frac{1}{n}\right)^n$ approaches as $n\to\infty$. In calculus, this is expressed using limit notation as $e=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$.

Examples:

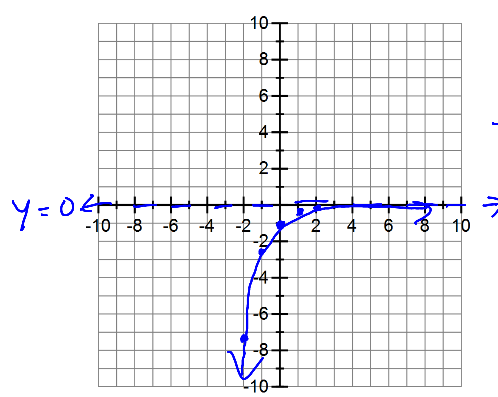
a) Graph $f(x)=e^x$.



x	y
-2	$e^{-2}\approx .1$
-1	$e^{-1}\approx .4$
0	$e^{0}=1$
1	$e^{1}\approx 2.7$
2	$e^{2}\approx 7.4$

b) Graph $f(x)=-e^{-x}$

reflect over both axes



x	y
-2	$-e^{-(-2)}\approx -7.4$
-1	$-e^{-(-1)}\approx -2.7$
0	$-e^{0}=-1$
1	$-e^{-1}\approx -.4$
2	$-e^{-2}\approx -.1$

Solving Exponential Equations

If $a > 0$ and $a \neq 1$ and $a^u = a^v$, then $u = v$.

Many exponential equations can be rewritten so the two sides have a common base. This allows us to set the exponents equal to each other and solve the equation.

Examples: Solve the following equations.

a) $3^{-x} = 243$

$3^{-x} = 3^5$

$-x = 5$

$\boxed{x = -5}$

b) $5^{x+3} = \dfrac{1}{5}$

$5^{x+3} = 5^{-1}$

$x + 3 = -1$

$\boxed{x = -4}$

c) $4^{x^2} = 2^x$

power to a power → multiply exponents

$(2^2)^{x^2} = 2^x$

$2^{2x^2} = 2^x$

$2x^2 = x$

$2x^2 - x = 0$

$x(2x-1) = 0$

$\boxed{x = 0}$ $\boxed{x = \frac{1}{2}}$

d) $3^{x^2-5x} = \dfrac{1}{81}$

$3^{x^2-5x} = 3^{-4}$

$x^2 - 5x = -4$

$x^2 - 5x + 4 = 0$

$(x-4)(x-1) = 0$

$\boxed{x = 4}$ $\boxed{x = 1}$

e) $4^x \cdot 2^{x^2} = 16^2$

$(2^2)^x \cdot 2^{x^2} = (2^4)^2$

$2^{2x} \cdot 2^{x^2} = 2^8$

add exponents

$2^{2x+x^2} = 2^8$

$2x + x^2 = 8$

$x^2 + 2x - 8 = 0$

$(x+4)(x-2) = 0$

$\boxed{x = -4}$ $\boxed{x = 2}$

f) $e^{x^2} = e^{3x} \cdot \dfrac{1}{e^2}$

$e^{x^2} = e^{3x} \cdot e^{-2}$

$e^{x^2} = e^{3x-2}$

$x^2 = 3x - 2$

$x^2 - 3x + 2 = 0$

$(x-2)(x-1) = 0$

$\boxed{x = 2}$ $\boxed{x = 1}$

$2^2 = 4$
$2^3 = 8$
$2^4 = 16$
$2^5 = 32$
$2^6 = 64$

$3^2 = 9$
$3^3 = 27$
$3^4 = 81$
$3^5 = 243$

$4^2 = 16$
$4^3 = 64$

$5^2 = 25$
$5^3 = 125$