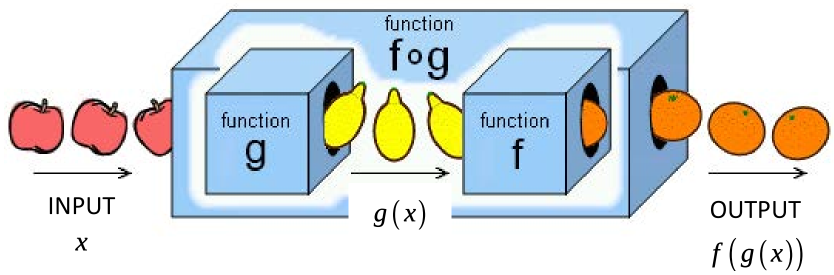
Composite Functions, Inverse Functions

Composite Function: In a composite function, one function is performed, and then a second function is performed on the result of the first function. $(f \circ g)(x) = f(g(x))$ and $(g \circ f)(x) = g(f(x))$.

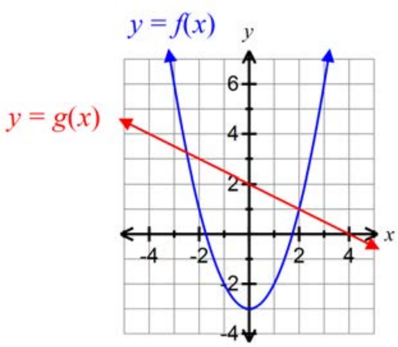


Hints:

- Work inside out. Plug the input into the inside function, then plug the result into the outside function.
- $(f \circ g)(x) = f(g(x))$ is not the same as $(f \cdot g)(x) = f(x) \cdot g(x)$.

Composition of functions — Multiplication of functions

Example: Evaluate each expression using the graph.



a) $(f \circ g)(4)$

$g(4) = 0$

$f(0) = \boxed{-3}$

b) $(g \circ f)(-1)$

$f(-1) = -2$

$g(-2) = \boxed{3}$

c) $(f \circ f)(1)$

$f(1) = -2$

$f(-2) = \boxed{1}$

d) $(g \circ g)(0)$

$g(0) = 2$

$g(2) = \boxed{1}$

Example: $f(x) = 2x^2$ and $g(x) = 1 - 3x^2$

a) Find $(f \circ g)(4)$

$g(4) = 1 - 3(4)^2 = -47$

$f(-47) = 2(-47)^2 = \boxed{4418}$

b) Find $(g \circ f)(2)$

$f(2) = 2(2)^2 = 8$

$g(8) = 1 - 3(8)^2 = \boxed{-191}$

c) Find $(f \circ f)(1)$

$f(1) = 2(1)^2 = 2$

$f(2) = 2(2)^2 = \boxed{8}$

d) Find $(g \circ g)(0)$

$g(0) = 1 - 3(0)^2 = 1$

$g(1) = 1 - 3(1)^2 = \boxed{-2}$

Domain of a Composite Function

The domain of $f \circ g$ is the set of all numbers x in the domain of g such that $g(x)$ is in the domain of f.

Examples: Find the domain of the composite function $(f \circ g)(x)$.

a) $f(x)=\dfrac{1}{x-2},\ g(x)=\sqrt{x}$

$x \to \boxed{\sqrt{x}} \to g(x) \to \boxed{\dfrac{1}{x-2}} \to f(g(x))$

$x \geq 0$

$g(x) \neq 2$

$\sqrt{x} \neq 2$

$x \neq 4$

$\boxed{\{x \mid x \geq 0, x \neq 4\} \text{ or } [0,4)\cup(4,\infty)}$

b) $f(x)=\dfrac{x}{x-1},\ g(x)=\dfrac{x+5}{x-4}$

$x \to \boxed{\dfrac{x+5}{x-4}} \to g(x) \to \boxed{\dfrac{x}{x-1}} \to f(g(x))$

$x \neq 4$

$g(x) \neq 1$

$\dfrac{x+5}{x-4} \neq 1$

$x+5 \neq x-4$

$\;\;-x \qquad -x$

$5 \neq -4$

This fraction is never equal to 1 (impossible to get 1 as output from first part of process) → no more domain restrictions

$\boxed{\{x \mid x \neq 4\} \text{ or } (-\infty,4)\cup(4,\infty)}$

Example: $f(x)=x+1$ and $g(x)=x^2+4$

a) Find $(f \circ g)(x)$ and its domain.

$f(g(x)) = f(x^2+4) = x^2+4+1$

$= \boxed{x^2+5}$

$D: \boxed{(-\infty,\infty)}$

b) Find $(g \circ f)(x)$ and its domain.

$g(f(x)) = g(x+1) = (x+1)^2+4$

$= x^2+2x+1+4$

$= \boxed{x^2+2x+5}$

$D: \boxed{(-\infty,\infty)}$

$(x+1)^2$

$=(x+1)(x+1)$

$=x^2+x+x+1$

$=x^2+2x+1$

Example: $f(x)=\dfrac{1}{x+3}$ and $g(x)=-\dfrac{2}{x}$

a) Find $(f \circ g)(x)$ and its domain.

$f\left(-\dfrac{2}{x}\right) = \dfrac{1}{-\frac{2}{x}+3} = \left(\dfrac{1}{-\frac{2}{x}+3}\right)\left(\dfrac{x}{x}\right)$

$x \neq 0$

$= \boxed{\dfrac{x}{-2+3x}}$

$-2+3x \neq 0$

$3x \neq 2$

$x \neq \frac{2}{3}$

$\boxed{D: \{x \mid x \neq 0, \tfrac{2}{3}\} \text{ or } (-\infty,0)\cup(0,\tfrac{2}{3})\cup(\tfrac{2}{3},\infty)}$

b) Find $(g \circ f)(x)$ and its domain.

$g\left(\dfrac{1}{x+3}\right) = \dfrac{-2}{\frac{1}{x+3}} = -2\left(\dfrac{x+3}{1}\right)$

$x \neq -3$

$= \boxed{-2x-6}$ ← no further domain restrictions

$\boxed{D: \{x \mid x \neq -3\} \text{ or } (-\infty,-3)\cup(-3,\infty)}$

c) Find $(f \circ f)(x)$ and its domain.

$f\left(\dfrac{1}{x+3}\right) = \dfrac{1}{\frac{1}{x+3}+3}$

$x \neq -3$

$= \left(\dfrac{1}{\frac{1}{x+3}+3}\right)\left(\dfrac{x+3}{x+3}\right)$

$= \dfrac{x+3}{1+3(x+3)} = \dfrac{x+3}{1+3x+9}$

$3x+10 \neq 0$

$3x \neq -10$

$x \neq -10/3$

$= \boxed{\dfrac{x+3}{3x+10}}$

$\boxed{D: \{x \mid x \neq -10/3, -3\} \text{ or } (-\infty,-10/3)\cup(-10/3,-3)\cup(-3,\infty)}$

d) Find $(g \circ g)(x)$ and its domain.

$g\left(-\dfrac{2}{x}\right) = \dfrac{-2}{-\frac{2}{x}} = -2\left(-\dfrac{x}{2}\right) = \boxed{x}$

$x \neq 0$

no further domain restrictions

$\boxed{D: \{x \mid x \neq 0\} \text{ or } (-\infty,0)\cup(0,\infty)}$

Example: Find functions f and g such that $f \circ g = H$.

a) $H(x)=(\underbrace{x^2+1}_{g(x)})^4$ outside = f(x)

$x \to \boxed{g(x) = x^2+1} \to g(x) \to \boxed{f(x) = x^4} \to f(g(x))$

$(x^2+1)^4$

b) $H(x)=|\underbrace{2x+1}_{g(x)}|$ outside = f(x)

$x \to \boxed{g(x) = 2x+1} \to g(x) \to \boxed{f(x) = |x|} \to f(g(x))$

$|2x+1|$

One-to-one Function: A function is ***one-to-one*** if for any value of x there is exactly one y (otherwise it wouldn't be a function), and for any value of y, there is exactly one x.

Example: Determine whether the following functions are one to one.

a) $\{(-2,6),(-1,3),(0,2),(1,5),(2,8)\}$

one-to-one
each x has exactly one y
each y has exactly one x

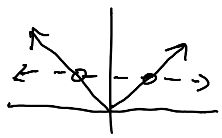
b) $\{(1,1),(2,4),(3,9),(0,0),(-1,1),(-2,4)\}$

function (each x has exactly one y)
not one-to-one (some y's have more than one x)

Horizontal Line Test: If every horizontal line intersects the graph of a function f in at most one point, then f is one-to-one.

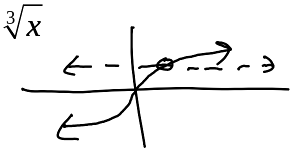
Example: For each function, use its graph to determine whether the function is one-to-one.

a) $f(x)=|x|$



not one-to-one

b) $g(x)=\sqrt[3]{x}$

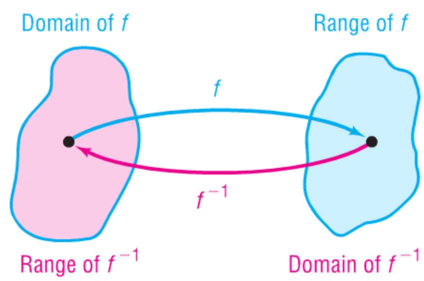


one-to-one

Theorem: A function that is increasing on an interval I is a one-to-one function on I.
A function that is decreasing on an interval I is a one-to-one function on I.

Inverse Function: Two functions are ***inverses*** if and only if whenever one function contains the element (a,b), the other function contains the element (b,a). If f is a one-to-one function, the correspondence from the range of f back to the domain of f is called the ***inverse function*** of f. The inverse of f is abbreviated f^{-1}.

Main idea of inverses:
Swap x & y



★ **Domain of f = Range of f^{-1}** **Range of f = Domain of f^{-1}**

Example: Find the inverse of the following one-to-one function: $\{(2,3),(4,5),(6,8),(9,10),(12,14)\}$

swap x's & y's: $\{(3,2),(5,4),(8,6),(10,9),(14,12)\}$

If we start with *x*, apply *f*, and then apply f^{-1}, we get *x* back again.
If we start with *x*, apply f^{-1}, and then apply *f*, we get *x* back again.
What f does, f^{-1} undoes, and vice versa. In other words,

$f^{-1}(f(x))=x$, where *x* is in the domain of f.

$f(f^{-1}(x))=x$ where *x* is in the domain of f^{-1}.

To verify that two functions are inverses, show that $f(g(x))=x$ and $g(f(x))=x$

Example: Verify that the functions are inverses.

a) $f(x)=4x$; $g(x)=x/4$

$f(g(x)) = 4\left(\frac{x}{4}\right) = x$ ✓

$g(f(x)) = \frac{4x}{4} = x$ ✓

b) $f(x)=4-3x$; $g(x)=\frac{1}{3}(4-x)$

$f(g(x)) = 4-3\left(\frac{1}{3}(4-x)\right)$
$= 4-3\left(\frac{4}{3}-\frac{1}{3}x\right) = 4-4+x = x$ ✓

$g(f(x)) = \frac{1}{3}(4-(4-3x))$
$= \frac{1}{3}(4-4+3x) = \frac{1}{3}(3x) = x$ ✓

c) $f(x)=\dfrac{2}{x+5}$; $g(x)=\dfrac{2}{x}-5$

$f(g(x)) = \dfrac{2}{\frac{2}{x}-5+5} = \dfrac{2}{\frac{2}{x}} = 2\left(\dfrac{x}{2}\right) = x$ ✓

$g(f(x)) = \dfrac{2}{\frac{2}{x+5}} - 5 = 2\left(\dfrac{x+5}{2}\right) - 5$

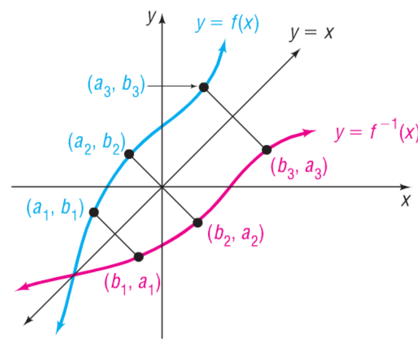
$= x+5-5 = x$ ✓

d) $f(x)=\sqrt[3]{2x}$; $g(x)=\dfrac{x^3}{2}$

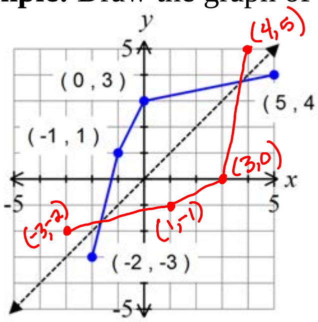
$f(g(x)) = \sqrt[3]{2\left(\dfrac{x^3}{2}\right)} = \sqrt[3]{x^3} = x$ ✓

$g(f(x)) = \dfrac{\left(\sqrt[3]{2x}\right)^3}{2} = \dfrac{2x}{2} = x$ ✓

Theorem: The graph of a function f and the graph of its inverse f^{-1} are symmetric with respect to the line $y = x$.



Example: Draw the graph of the inverse function.



swap x & y

Finding the Inverse of a Function

1. Rewrite $f(x)$ as y in the original equation.
2. Interchange x and y.
3. Solve for y.
4. Replace y with the notation $f^{-1}(x)$.

Example: Find the inverse. State the domain and range of $f(x)$ and the domain and range of $f^{-1}(x)$.

a) $f(x)=-3x+1$

Domain of f: $(-\infty,\infty)$
Range of f: $(-\infty,\infty)$
Domain of f^{-1}: $(-\infty,\infty)$
Range of f^{-1}: $(-\infty,\infty)$

Swap x&y: $x=-3y+1$

Solve for y: $\dfrac{x-1}{-3} = \dfrac{-3y}{-3}$

$y = \dfrac{x-1}{-3}$ or $-\dfrac{1}{3}x+\dfrac{1}{3}$

rename as $f^{-1}(x)$: $\boxed{f^{-1}(x) = \dfrac{x-1}{-3} \text{ or } f^{-1}(x) = -\dfrac{1}{3}x+\dfrac{1}{3}}$

b) $f(x)=\dfrac{2x+3}{5x-4}$

Domain of f: $\{x \mid x \neq \frac{4}{5}\}$
Range of f: $\{y \mid y \neq \frac{2}{5}\}$
Domain of f^{-1}: $\{x \mid x \neq \frac{2}{5}\}$
Range of f^{-1}: $\{y \mid y \neq \frac{4}{5}\}$

Find domains from equations then ranges by looking at domain of inverse.

swap x&y: $x = \dfrac{2y+3}{5y-4}$

multiply both sides by denom. to get rid of fraction: $x(5y-4)=2y+3$

distribute: $5xy-4x=2y+3$
$\quad -2y+4x \quad -2y+4x$

get everything w/ y on one side, everything else on other: $5xy-2y=4x+3$

factor out y: $y(5x-2)=4x+3$

divide: $y = \boxed{\dfrac{4x+3}{5x-2} = f^{-1}(x)}$