# The Graph of a Rational Function

**Analyzing the Graph of a Rational Function**

1. Completely factor the numerator and denominator.
2. List the key features of the graph.
   - **Domain:** Set the denominator of the ***unsimplified function*** not equal to zero and solve.
   - ***x*-intercept(s):** *S*et the numerator of the ***simplified function*** equal to zero and solve.
   - ***y*-intercept:** Plug in $x = 0$. (You can do this before or after simplifying the function, but remember, if zero is not in the domain, the function has no *y*-intercept.)
   - **Holes:** Find the *x*-coordinates by setting factors that cancel out equal to zero and solving. Find the *y*-coordinate by plugging the *x*-coordinate into the simplified function.
   - **Vertical Asymptote(s):** *S*et the denominator of the ***simplified function*** equal to zero and solve.
   - **Horizontal or Oblique Asymptote:**
     - If *degree of numerator < degree of denominator*, the graph has a **horizontal asymptote** at $y = 0$.
     - If *degree of numerator = degree of denominator*,the graph has a **horizontal asymptote** at the line $y = \dfrac{\text{leading coefficient of the numerator}}{\text{leading coefficient of the denominator}}$.
     - If *degree of numerator > degree of denominator, (by one),* the graph has an **oblique asymptote** of $y = mx + b,\ m \neq 0$ found by performing long or synthetic division.
     - If *degree of numerator > degree of denominator, (by more than one),* then *R* has **neither a horizontal nor an oblique asymptote**, but the end behavior can be determined using long or synthetic division.
3. Use the *x*-intercepts and vertical asymptotes as **critical points** to divide the graph into intervals. Plug in values of *x* on each side of these points to find out what the graph is doing in each interval.
4. Analyze the behavior of the graph near each asymptote.
5. Put together all of the information to graph the function.

**Example:** Analyze the Graph of the Rational Function $R(x) = \dfrac{x+2}{x^2 - 9} = \dfrac{x+2}{(x+3)(x-3)}$

Domain: $(x+3)(x-3) \neq 0$

$\{x \mid x \neq -3, 3\}$

x-int: $x + 2 = 0$

$x = -2$ $\quad (-2, 0)$

y-int: $R(0) = \dfrac{0+2}{0^2 - 9} = -\dfrac{2}{9}$ $\quad \left(0, -\dfrac{2}{9}\right)$

Holes: No factors canceled out

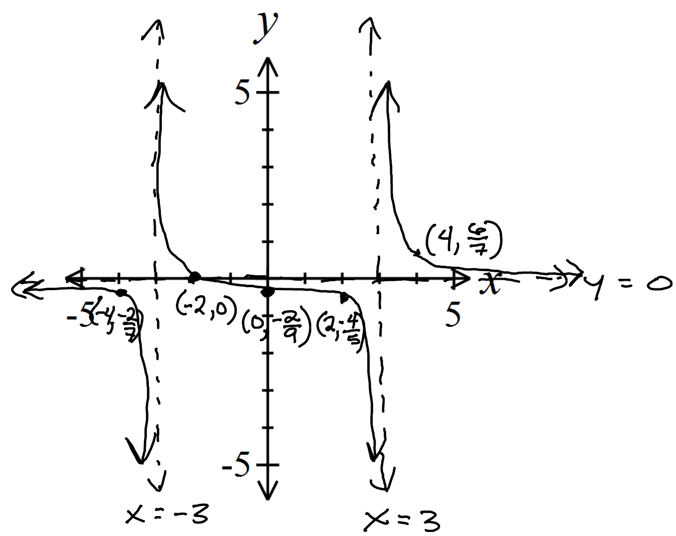
→ no holes

vert. asymps: $(x+3)(x-3) = 0$

$x = -3,\ x = 3$

horiz/oblique asymp: $\dfrac{\text{Deg } 1}{\text{Deg } 2}$

Horizontal asymp @ $y = 0$



$A^3+B^3=(A+B)(A^2-AB+B^2)$

$x^3+1^3=(x+1)(x^2-x\cdot 1+1^2)$

**Example:** Analyze the Graph of the Rational Function $R(x)=\dfrac{x^3+1}{x^2+2x}=\dfrac{(x+1)(x^2-x+1)}{x(x+2)}$

Domain: $x(x+2)\neq 0$ $\quad \{x \mid x\neq 0,-2\}$

x-int: $(x+1)(x^2-x+1)=0$

$x=-1$ $\quad$ imag. zeros $\quad (-1,0)$

y-int: 0 not in domain. $R(0)$ undefined. no y-int.

holes: no factors cancelled out → no holes

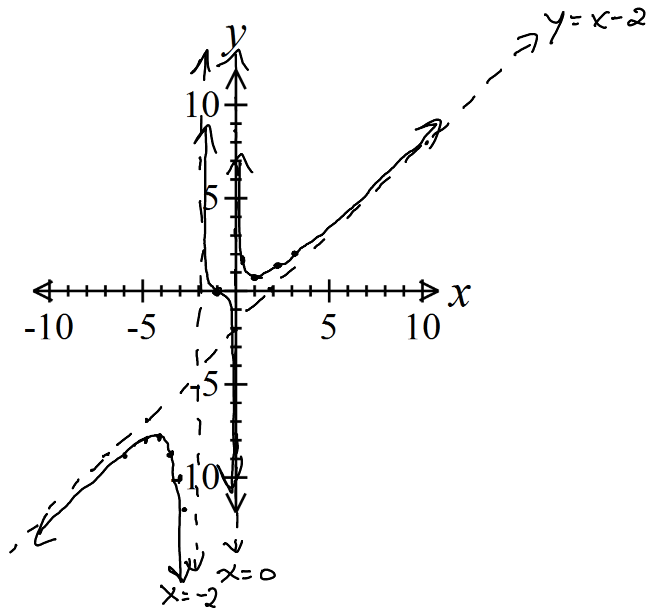
Vert. asymps: $x(x+2)=0 \rightarrow x=0, x=-2$

horiz/oblique asymp. $\dfrac{\text{Deg } 3}{\text{Deg } 2}$ deg num > deg denom by 1

oblique asymptote

$y=x-2$

$$\begin{array}{r} x-2 \\ x^2+2x\overline{)\,x^3+0x^2+0x+1} \\ -(x^3+2x^2) \\ \hline -2x^2+0x \\ -(-2x^2-4x) \\ \hline 4x+1 \end{array}$$




**Example:** Analyze the Graph of the Rational Function $R(x)=\dfrac{x^4+x^2+1}{x^2-1}$. ← Top: prime. u-substitution w/ $u=x^2$: $u^2+u+1$ is prime.

$R(x)=\dfrac{x^4+x^2+1}{(x+1)(x-1)}$

Domain: $(x+1)(x-1)\neq 0$

$\{x \mid x\neq -1,1\}$

x-int: $x^4+x^2+1=0$ $\quad u=x^2$

$u^2+u+1=0$

$u=\dfrac{-1\pm\sqrt{1^2-4(1)(1)}}{2(1)}=\dfrac{-1\pm\sqrt{-3}}{2}$ ← imag.

no x-ints

y-int: $R(0)=\dfrac{0^4+0^2+1}{0^2-1}=-1$ $\quad (0,-1)$

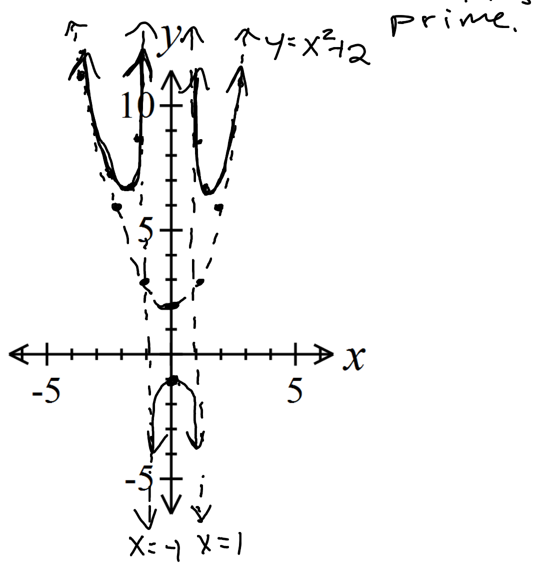
holes: nothing canceled out → no holes

V. asyms: $(x+1)(x-1)=0$

$x=-1, x=1$

horiz/oblique: $\dfrac{\text{Deg } 4}{\text{Deg } 2}$ top higher by 2. Ends approach parabola.

Ends approach $y=x^2+2$




$$\begin{array}{r} x^2+2 \\ x^2-1\overline{)\,x^4+0x^3+x^2+0x+1} \\ -(x^4\qquad -x^2) \\ \hline +2x^2+0x+1 \\ -(2x^2\qquad -2) \\ \hline 1 \end{array}$$

**Example:** Analyze the Graph of the Rational Function $R(x)=\frac{x^2+x-12}{x^2-4}=\frac{(x+4)(x-3)}{(x+2)(x-2)}$

Domain: $(x+2)(x-2)\neq 0$

$\{x \mid x\neq -2, 2\}$

x-ints: $(x+4)(x-3)=0$

$x=-4,\ x=3$ $\quad (-4,0)$ $(3,0)$

y-int: $R(0)=\frac{0^2+0-12}{0^2-4}=3$ $\quad (0,3)$

holes: No factors canceled → no holes

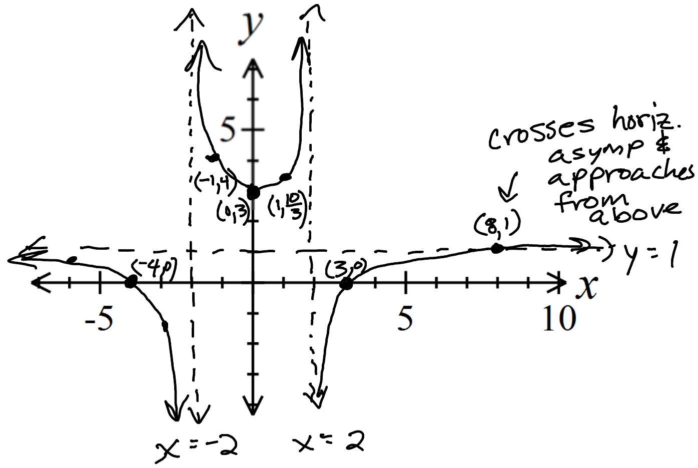
vert. asymps: $(x+2)(x-2)=0$

$x=-2 \quad x=2$

horiz/oblique asymp: $\frac{\text{deg } 2}{\text{deg } 2}$

horiz @ $y=\frac{1}{1}$

$y=1$




**Example:** Analyze the Graph of the Rational Function $R(x)=\frac{x^2+3x-10}{x^2+8x+15}=\frac{\cancel{(x+5)}(x-2)}{\cancel{(x+5)}(x+3)}=\frac{x-2}{x+3}$

hole @ $x=-5$

Domain: $(x+5)(x+3)\neq 0$

$\{x \mid x\neq -5, -3\}$

hole $\quad$ v. asymp

x-int: $x-2=0 \Rightarrow x=2$ $\quad (2,0)$

y-int: $R(0)=\frac{0-2}{0+3}=-\frac{2}{3}$ $\quad \left(0,-\frac{2}{3}\right)$

hole: $x+5=0$

$x=-5$

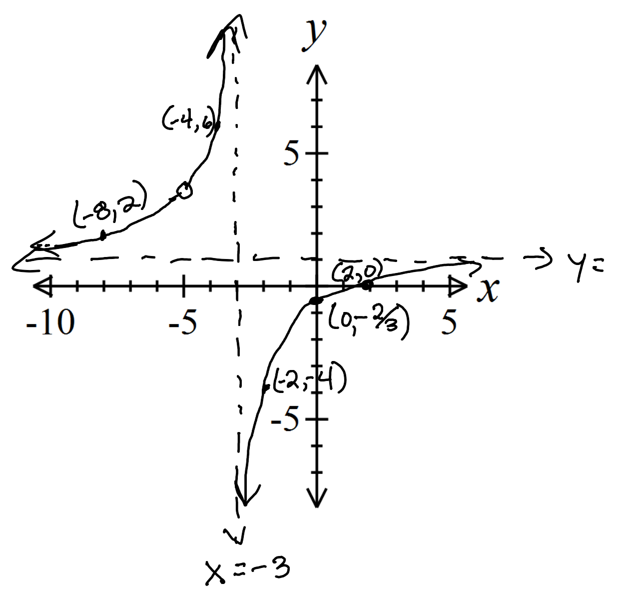
Plug -5 into simplified function to get y: $\frac{-5-2}{-5+3}=\frac{-7}{-2}=\frac{7}{2}$ $\quad \left(-5,\frac{7}{2}\right)$

v. asymp: $x+3=0$

$x=-3$

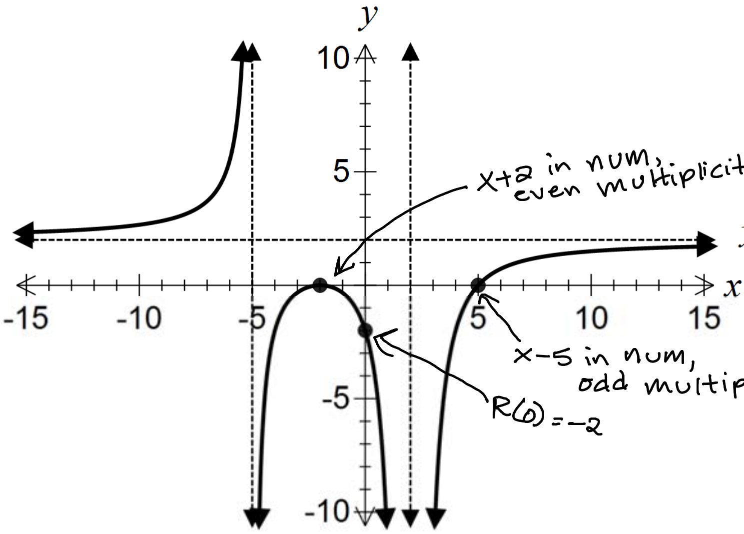
horiz/oblique asymp: $\frac{\text{deg } 1}{\text{deg } 1}$

$y=\frac{1}{1}$ $\quad$ horizontal @ $y=1$



**Example:** Find a rational function that might have the graph shown below.

★ **Note:** If the graph goes the same direction on both sides of an asymptote (approaches $\infty$ on both sides or approaches $-\infty$ on both sides), the related factor in the denominator has an even multiplicity. If the graph goes in opposite directions on the two sides of an asymptote (approaches $\infty$ on one side and $-\infty$ on the other), the related factor in the denominator has an odd multiplicity.




x+2 in num, even multiplicity (touches)

numerator & denominator are same degree & ratio of leading coefficients is 2

x−5 in num, odd multiplicity (crosses)

R(0) = −2

$x = -5$
up on left, down on right
x+5 in denom, odd multiplicity (opposite directions on two sides of asymptote)

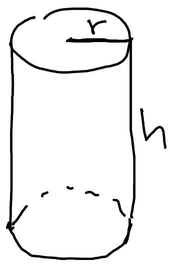
$x = 2$
down on both sides
x−2 in denom, even multiplicity (same direction on both sides of asymptote)

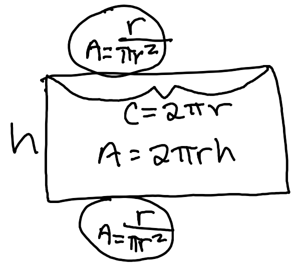
$$\boxed{R(x) = \frac{2(x-5)(x+2)^2}{(x+5)(x-2)^2}}$$

**Example:** Finding the Least Cost of a Can

Reynolds Metal Company manufactures aluminum cans in the shape of a right circular cylinder with a capacity of 500 cubic centimeters. The top and bottom of the can are made of special aluminum alloy that costs 0.05¢ per square centimeter. The sides of the cans are made of material that costs 0.02¢ per square centimeter.

a) Express the cost *C* of material for the can as a function of the radius *r* of the can.




Combined area of top & bottom: $2\pi r^2$ cm$^2$

Area of sides: $2\pi rh$ cm$^2$

$$\frac{V}{\pi r^2} = \frac{\pi r^2 h}{\pi r^2} = \frac{500 \text{ cm}^3}{\pi r^2}$$

$$h = \frac{500}{\pi r^2}$$

$$\text{Cost} = (2\pi r^2)(0.05) + (2\pi rh)(0.02)$$

$$\text{Cost} = 0.1\pi r^2 + 0.04\pi rh$$

$$C(r) = 0.1\pi r^2 + 0.04\pi r\left(\frac{500}{\pi r^2}\right)$$

$$\boxed{C(r) = 0.1\pi r^2 + \frac{20}{r} \text{ cents}}$$

b) Find any vertical asymptotes. Discuss the cost of the can near any vertical asymptotes.

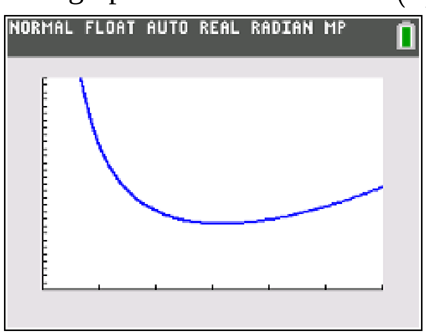
$$C(r) = 0.1\pi r^2\left(\frac{r}{r}\right) + \frac{20}{r}$$

$$C(r) = \frac{0.1\pi r^3 + 20}{r}$$

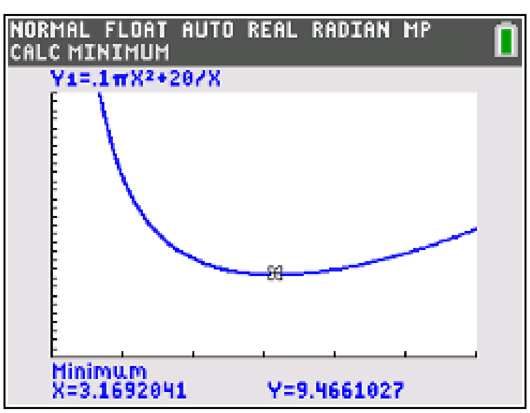
$\boxed{\text{Vert. asymp: } r = 0}$

The cost becomes extremely high as the radius approaches 0 because the height would have to increase to keep the volume of the jar constant, which would increase the cost of the materials for the sides of the can.

c) Use a graphing calculator to graph the function $C = C(r)$.




d) What value of *r* will result in the least cost? What is the least cost?




A radius of about 3.2 cm will result in the least cost. The least cost is 9.5¢.