Properties of Rational Functions

A ***rational function*** is a function of the form $R(x)=\dfrac{p(x)}{q(x)}$, where p and q are polynomial functions and q is not the zero polynomial.

Finding the Domain of a Rational Function

The domain of a rational function is the set of all real numbers except those for which the denominator q is 0.

Examples: Find the domain of each rational function.

a) $R(x)=\dfrac{2x^2-3}{x+3}$

b) $R(x)=\dfrac{x+3}{x^2-9}$

c) $R(x)=\dfrac{x^2}{x^2+7x+12}$

★ **Note:** It is important to understand that $R(x)=\dfrac{x+3}{x^2-9}$ and $f(x)=\dfrac{1}{x-3}$ are not the same. They have different domains. Their graphs are nearly identical, but the graph of the first function has a hole in it at $x=-3$, while the graph of the second function does not.

A rational function $R(x)=\dfrac{p(x)}{q(x)}$ is in ***lowest terms*** if $p(x)$ and $q(x)$ have no common factors.

Finding the Intercepts of the Graph of a Rational Function

★ **Simplify before finding the *x*-intercepts of the graph!** Otherwise, you may end up listing values that are actually holes in the graph.

★ When finding the *y*-intercepts, remember that **if zero is not in the domain, there is no *y*-intercept!**

To find the *x*-intercepts (real zeros) of the graph of a rational function, we set $R(x)=0$ and solve for x.

Notice that if $R(x)=\dfrac{p(x)}{q(x)}=0$, $p(x)$ must equal zero. It is the numerator that tells us about the *x*-intercepts.

- To find the ***x*-intercepts** (real zeros), simplify the rational function and set the numerator equal to zero.
- To find the ***y*-intercept,** make sure 0 is in the domain, then plug $x=0$ into either the simplified or the unsimplified function.

Examples: Find the *x*- and *y*- intercepts of each rational function.

a) $R(x)=\dfrac{x+2}{x^2-16}$

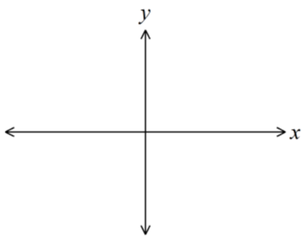
b) $R(x)=\dfrac{x^2-2x-35}{x^2+11x+30}$

c) $R(x)=\dfrac{x}{x^2+8x}$

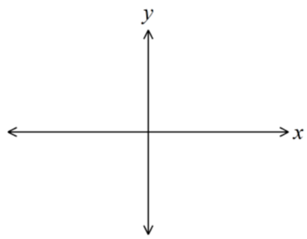
Asymptotes

Let R denote a function.

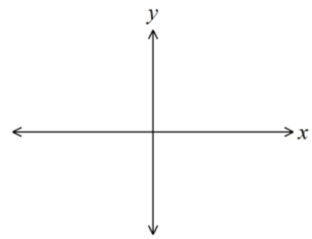
- If, as $x \to -\infty$ or as $x \to \infty$, the values of $R(x)$ approach some fixed number L, then the line $y = L$ is a ***horizontal asymptote*** of the graph of R.
- If, as x approaches some number c, the values of $R(x)$ approach ∞ or $-\infty$, then the line $x = c$ is a ***vertical asymptote*** of the graph of R.



$\lim_{x \to \infty} R(x) = L$

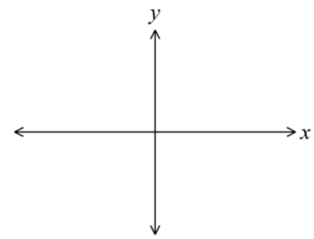


$\lim_{x \to -\infty} R(x) = L$



$\lim_{x \to c^-} R(x) = \infty$

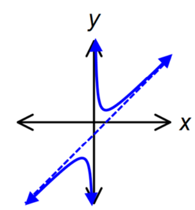
$\lim_{x \to c^+} R(x) = \infty$



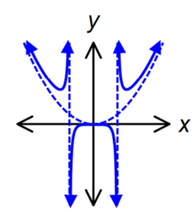
$\lim_{x \to c^-} R(x) = -\infty$

$\lim_{x \to c^+} R(x) = \infty$

There is a third type of asymptote called an ***oblique asymptote*****.** An oblique asymptote occurs when the graph's end behavior follows an oblique line (a linear function of the form $y = mx + b,\ m \neq 0$).



A graph may also approach some function such as a parabola, but since asymptotes are defined as straight lines, we do not refer to these as asymptotes.



★ **Note:** The graph of a function may intersect a horizontal or oblique asymptote at some other point, but the graph will never intersect a vertical asymptote.

Finding the Vertical Asymptotes of a Rational Function

A rational function $R(x) = \dfrac{p(x)}{q(x)}$, in *lowest terms* will have a vertical asymptote at $x = r$ if r is a real zero of the denominator q. That is, if $x - r$ is a factor of the denominator q of the rational function in lowest terms, it will have the vertical asymptote $x = r$.

★ **Simplify before finding the vertical asymptotes!** If the rational function is not in lowest terms, this theorem will result in an incorrect listing of vertical asymptotes.

Example: Find the domain and the vertical asymptotes, if any, of the graph of each rational function.

a) $R(x) = \dfrac{x+1}{x^2 - 9}$

b) $R(x) = \dfrac{x+2}{x^2 - 3x - 10}$

c) $R(x) = \dfrac{x}{x^3 + 10x^2 + 9x}$

d) $R(x) = \dfrac{x^2}{x^2 + 4}$

★ **Note:** The graph of a rational function can have several vertical asymptotes, but it will never have multiple horizontal or oblique asymptotes, and it cannot have both a horizontal and an oblique asymptote. It will either have one horizontal asymptote, one oblique asymptote, or neither.

Find the Horizontal or Oblique Asymptotes of a Rational Function

- If *degree of numerator* < *degree of denominator,* the graph has a **horizontal asymptote** at $y = 0$.
- If *degree of numerator* = *degree of denominator,* the graph has a **horizontal asymptote** at the line $y = \dfrac{\text{leading coefficient of the numerator}}{\text{leading coefficient of the denominator}}$.
- If *degree of numerator* > *degree of denominator (by one)*, the graph has an **oblique asymptote** of $y = mx + b$ found by performing long or synthetic division.
- If *degree of numerator* > *degree of denominator (by more than one)*, then *R* has neither a horizontal nor an oblique asymptote, but the end behavior can be determined using long or synthetic division.

Examples: Find the horizontal or oblique asymptotes, if any, of the graph of the function.

a) $R(x) = \dfrac{x+5}{x^2 - 3x - 10}$

b) $H(x) = \dfrac{6x^2 - 2x + 5}{3x^2 - 2}$

c) $H(x) = \dfrac{8x^3 + 2x^2 - 6}{2x^2 - 3}$

d) $H(x) = \dfrac{3x^4 + 4x^2 - 7}{x^2 - 2x + 5}$

Summary:

Given a rational function $R(x) = \frac{p(x)}{q(x)}$,

1. **Domain:** Set the denominator of the ***unsimplified function*** not equal to zero and solve.
2. ***x*-intercept(s):** Set the numerator of the ***simplified function*** equal to zero and solve.
3. ***y*-intercept:** Plug in $x = 0$. (You can do this before or after simplifying the function, but remember, if zero is not in the domain, the function has no *y*-intercept.)
4. **Holes:** Find the *x*-coordinates by setting factors that cancel out equal to zero and solving. Find the *y*-coordinate by plugging the *x*-coordinate into the simplified function.
5. **Vertical Asymptote(s):** Set the denominator of the ***simplified function*** equal to zero and solve.
6. **Horizontal or Oblique Asymptote:**
   - If *degree of numerator* < *degree of denominator,* the graph has a **horizontal asymptote** at $y = 0$.
   - If *degree of numerator* = *degree of denominator,* the graph has a **horizontal asymptote** at the line $y = \frac{\text{leading coefficient of the numerator}}{\text{leading coefficient of the denominator}}$.
   - If *degree of numerator* > *degree of denominator (by one)*, the graph has an **oblique asymptote** of $y = mx + b$ found by performing long or synthetic division.
   - If *degree of numerator* > *degree of denominator (by more than one)*, then *R* has neither a horizontal nor an oblique asymptote, but the end behavior can be determined using long or synthetic division.