Properties of Rational Functions

A ***rational function*** is a function of the form $R(x)=\dfrac{p(x)}{q(x)}$, where *p* and *q* are polynomial functions and *q* is not the zero polynomial.

Finding the Domain of a Rational Function

The domain of a rational function is the set of all real numbers except those for which the denominator *q* is 0.

Examples: Find the domain of each rational function.

a) $R(x)=\dfrac{2x^2-3}{x+3}$

$x+3\neq 0$

$\boxed{\{x \mid x\neq -3\}}$

b) $R(x)=\dfrac{x+3}{x^2-9}=\dfrac{x+3}{(x+3)(x-3)}$

Find domain before simplifying

$(x+3)(x-3)\neq 0$

$\boxed{\{x \mid x\neq -3, 3\}}$

c) $R(x)=\dfrac{x^2}{x^2+7x+12}=\dfrac{x^2}{(x+3)(x+4)}$

$(x+3)(x+4)\neq 0$

$\boxed{\{x \mid x\neq -3, -4\}}$

★ **Note:** It is important to understand that $R(x)=\dfrac{x+3}{x^2-9}$ and $f(x)=\dfrac{1}{x-3}$ are not the same. They have different domains. Their graphs are nearly identical, but the graph of the first function has a hole in it at $x=-3$, while the graph of the second function does not.

A rational function $R(x)=\dfrac{p(x)}{q(x)}$ is in ***lowest terms*** if $p(x)$ and $q(x)$ have no common factors.

Finding the Intercepts of the Graph of a Rational Function

★ **Simplify before finding the *x*-intercepts of the graph!** Otherwise, you may end up listing values that are actually holes in the graph.

★ When finding the *y*-intercepts, remember that **if zero is not in the domain, there is no *y*-intercept!**

To find the *x*-intercepts (real zeros) of the graph of a rational function, we set $R(x)=0$ and solve for x.

Notice that if $R(x)=\dfrac{p(x)}{q(x)}=0$, $p(x)$ must equal zero. It is the numerator that tells us about the *x*-intercepts.

- To find the ***x*-intercepts** (real zeros), simplify the rational function and set the numerator equal to zero.
- To find the ***y*-intercept,** make sure 0 is in the domain, then plug $x=0$ into either the simplified or the unsimplified function.

Examples: Find the *x*- and *y*- intercepts of each rational function.

a) $R(x)=\dfrac{x+2}{x^2-16}=\dfrac{x+2}{(x+4)(x-4)}$

x-int: $x+2=0$, $x=-2$ $\boxed{(-2,0)}$

y-int: $R(0)=\dfrac{0+2}{0^2-16}=\dfrac{2}{-16}=-\dfrac{1}{8}$

$\boxed{(0,-\tfrac{1}{8})}$

b) $R(x)=\dfrac{x^2-2x-35}{x^2+11x+30}=\dfrac{(x-7)\cancel{(x+5)}}{(x+6)\cancel{(x+5)}}$

Hole @ $x=-5$
Not an x-int

$=\dfrac{x-7}{x+6}$

x-int: $x-7=0$, $x=7$ $\boxed{(7,0)}$

y-int: $R(0)=\dfrac{0-7}{0+6}=\dfrac{-7}{6}$

$\boxed{(0,-\tfrac{7}{6})}$

c) $R(x)=\dfrac{x}{x^2+8x}=\dfrac{\cancel{x}}{\cancel{x}(x+8)}$

Hole @ $x=0$

$=\dfrac{1}{x+8}$

Numerator of simplified function can never equal zero

$\boxed{\text{No x-int}}$

Domain: $\{x \mid x\neq 0,-8\}$

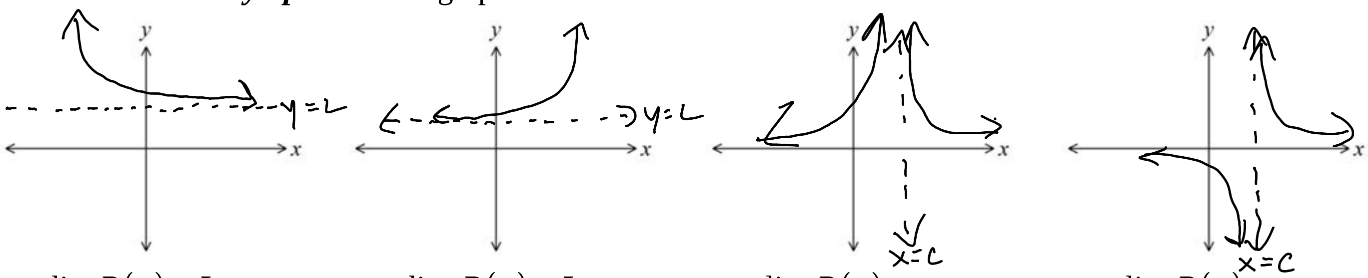
0 not in domain!

$\boxed{\text{No y-int}}$

Asymptotes

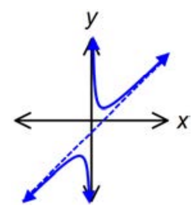
Let R denote a function.

- If, as $x \to -\infty$ or as $x \to \infty$, the values of $R(x)$ approach some fixed number L, then the line $y = L$ is a ***horizontal asymptote*** of the graph of R.
- If, as x approaches some number c, the values of $R(x)$ approach ∞ or $-\infty$, then the line $x = c$ is a ***vertical asymptote*** of the graph of R.

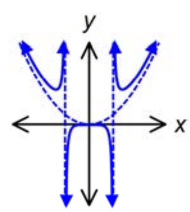


$\lim\limits_{x\to\infty} R(x) = L$ $\qquad$ $\lim\limits_{x\to-\infty} R(x) = L$ $\qquad$ $\lim\limits_{x\to c^-} R(x) = \infty$, $\lim\limits_{x\to c^+} R(x) = \infty$ $\qquad$ $\lim\limits_{x\to c^-} R(x) = -\infty$, $\lim\limits_{x\to c^+} R(x) = \infty$

There is a third type of asymptote called an ***oblique asymptote*.** An oblique asymptote occurs when the graph's end behavior follows an oblique line (a linear function of the form $y = mx + b,\ m \neq 0$).



A graph may also approach some function such as a parabola, but since asymptotes are defined as straight lines, we do not refer to these as asymptotes.



★ **Note:** The graph of a function may intersect a horizontal or oblique asymptote at some other point, but the graph will never intersect a vertical asymptote.

Finding the Vertical Asymptotes of a Rational Function

A rational function $R(x) = \dfrac{p(x)}{q(x)}$, in *lowest terms* will have a vertical asymptote at $x = r$ if r is a real zero of the denominator q. That is, if $x - r$ is a factor of the denominator q of the rational function in lowest terms, it will have the vertical asymptote $x = r$.

★ **Simplify before finding the vertical asymptotes!** If the rational function is not in lowest terms, this theorem will result in an incorrect listing of vertical asymptotes.

Example: Find the domain and the vertical asymptotes, if any, of the graph of each rational function.

a) $R(x) = \dfrac{x+1}{x^2-9} = \dfrac{x+1}{(x+3)(x-3)}$

Domain: $(x+3)(x-3) \neq 0$

$\boxed{\{x \mid x \neq -3, 3\}}$

Vert. asymps: $(x+3)(x-3) = 0$

$\boxed{x = -3,\ x = 3}$

b) $R(x) = \dfrac{x+2}{x^2-3x-10} = \dfrac{\cancel{x+2}}{(x-5)\cancel{(x+2)}}$

$= \dfrac{1}{x-5}$

Domain: $(x-5)(x+2) \neq 0$ ← unsimplified

$\boxed{\{x \mid x \neq 5, -2\}}$

Vert. asymps: $x - 5 = 0$ ← simplified

$\boxed{x = 5}$

(There is a hole w/ an x-coord of −2)

c) $R(x)=\dfrac{x}{x^3+10x^2+9x} = \dfrac{x}{x(x^2+10x+9)} = \dfrac{\cancel{x}}{\cancel{x}(x+9)(x+1)}$
$= \dfrac{1}{(x+9)(x+1)}$

Domain: $x(x+9)(x+1) \neq 0$ ← unsimplified
$\boxed{\{x \mid x \neq 0, -9, -1\}}$

Vert. asymps: $(x+9)(x+1)=0$ ← simplified
$\boxed{x=-9,\ x=-1}$

d) $R(x)=\dfrac{x^2}{x^2+4}$ ← prime

Domain: $x^2+4 \neq 0$
$x^2 \neq -4$
$x \neq \pm\sqrt{-4}$ ← imag.
$\boxed{\text{Domain is } \mathbb{R}}$
$\boxed{\text{No vertical asymptotes}}$

★ **Note:** The graph of a rational function can have several vertical asymptotes, but it will never have multiple horizontal or oblique asymptotes, and it cannot have both a horizontal and an oblique asymptote. It will either have one horizontal asymptote, one oblique asymptote, or neither.

Find the Horizontal or Oblique Asymptotes of a Rational Function

- If *degree of numerator* < *degree of denominator*, the graph has a **horizontal asymptote** at $y=0$.
- If *degree of numerator* = *degree of denominator*, the graph has a **horizontal asymptote** at the line $y=\dfrac{\text{leading coefficient of the numerator}}{\text{leading coefficient of the denominator}}$.
- If *degree of numerator* > *degree of denominator (by one)*, the graph has an **oblique asymptote** of $y=f(x)$ found by performing long or synthetic division.
- If *degree of numerator* > *degree of denominator (by more than one)*, then *R* has neither a horizontal nor an oblique asymptote, but the end behavior can be determined using long or synthetic division.

Examples: Find the horizontal or oblique asymptotes, if any, of the graph of the function.

a) $R(x)=\dfrac{x+5}{x^2-3x-10}$ $\dfrac{\text{Deg } 1}{\text{Deg } 2}$

$\boxed{\text{horizontal asymptote}}$
$\boxed{y=0}$

b) $H(x)=\dfrac{6x^2-2x+5}{3x^2-2}$ $\dfrac{\text{Deg } 2}{\text{Deg } 2}$

$\boxed{\text{horizontal asymptote}}$
$y=\frac{6}{3}$ $\boxed{y=2}$

c) $H(x)=\dfrac{8x^3+2x^2-6}{2x^2-3}$ $\dfrac{\text{Deg } 3}{\text{Deg } 2}$

$\boxed{\text{oblique asymptote}}$

$$
\begin{array}{r}
4x+1 \\
2x^2-3 \overline{\big)\, 8x^3+2x^2+0x-6} \\
-(8x^3 -12x) \\
\hline
2x^2+12x-6 \\
-(2x^2 -3) \\
\hline
12x-3
\end{array}
$$

$\boxed{y=4x+1}$

d) $H(x)=\dfrac{3x^4+4x^2-7}{x^2-2x+5}$ $\dfrac{\text{Deg } 4}{\text{Deg } 2}$

$\boxed{\text{neither horizontal nor oblique}}$

$$
\begin{array}{r}
3x^2+6x+1 \\
x^2-2x+5 \overline{\big)\, 3x^4+0x^3+4x^2+0x-7} \\
-(3x^4-6x^3+15x^2) \\
\hline
6x^3-11x^2+0x \\
-(6x^3-12x^2+30x) \\
\hline
x^2-30x-7 \\
-(x^2-2x+5) \\
\hline
28x-12
\end{array}
$$

$\boxed{\text{Ends approach } y=3x^2+6x+1}$

Summary:

Given a rational function $R(x) = \frac{p(x)}{q(x)}$,

1. **Domain:** Set the denominator of the ***unsimplified function*** not equal to zero and solve.
2. ***x*-intercept(s):** Set the numerator of the ***simplified function*** equal to zero and solve.
3. ***y*-intercept:** Plug in $x = 0$. (You can do this before or after simplifying the function, but remember, if zero is not in the domain, the function has no *y*-intercept.)
4. **Holes:** Find the *x*-coordinates by setting factors that cancel out equal to zero and solving. Find the *y*-coordinate by plugging the *x*-coordinate into the simplified function.
5. **Vertical Asymptote(s):** Set the denominator of the ***simplified function*** equal to zero and solve.
6. **Horizontal or Oblique Asymptote:**
   - If $\textit{degree of numerator} < \textit{degree of denominator}$, the graph has a **horizontal asymptote** at $y = 0$.
   - If $\textit{degree of numerator} = \textit{degree of denominator}$, the graph has a **horizontal asymptote** at the line $y = \frac{\text{leading coefficient of the numerator}}{\text{leading coefficient of the denominator}}$.
   - If $\textit{degree of numerator} > \textit{degree of denominator (by one)}$, the graph has an **oblique asymptote** of $y = mx + b,\ m \neq 0$ found by performing long or synthetic division.
   - If $\textit{degree of numerator} > \textit{degree of denominator (by more than one)}$, then R has **neither a horizontal nor an oblique asymptote**, but the end behavior can be determined using long or synthetic division.