The Zeros of Polynomial Functions

Remainder Theorem: If a polynomial function $f(x)$ is divided by $x-c$, then the remainder is $f(c)$.

Example: Find the remainder if $f(x)=2x^3-3x^2+1$ is divided by

a) $x+1$

$f(-1)=2(-1)^3-3(-1)^2+1$
$=2(-1)-3(1)+1$
$=-2-3+1=\boxed{-4}$

b) $x-3$

$f(3)=2(3)^3-3(3)^2+1$
$=2(27)-3(9)+1$
$=54-27+1=\boxed{28}$

Factor Theorem: If f is a polynomial function, then $x-c$ is a factor of $f(x)$ if and only if $f(c)=0.$ The Factor Theorem means that:

a) If $f(c)=0,$ then $x-c$ is a factor of $f(x)$.
b) If $x-c$ is a factor of $f(x)$, then $f(c)=0$.

Example: Use the factor theorem to determine whether the function $f(x)=4x^4-15x^2-4$ has the factor

a) $x-2$

$f(2)=4(2)^4-15(2)^2-4$
$=4(16)-15(4)-4$
$=64-60-4=0$

$\boxed{\text{factor}}$

b) $x+1$

$f(-1)=4(-1)^4-15(-1)^2-4$
$=4(1)-15(1)-4$
$=4-15-4=-15$

$\boxed{\text{not a factor}}$

Number of Real Zeros Theorem: A polynomial function cannot have more real zeros than its degree.

Rational Zeros Theorem: Let f be a polynomial function of degree 1 or higher, where each coefficient is an integer. The possible rational zeros are all numbers $\frac{p}{q}$, where p is a factor of the constant, and q is a factor of the leading coefficient of the function.

Example: List the potential rational zeros of $f(x)=3x^4+4x^3+7x^2+8x+2.$

Factors of constant (2): $\pm 1, \pm 2$ ← Top
Factors of leading coefficient (3): $\pm 1, \pm 3$ ← Bottom
$\boxed{\text{Potential rational zeros: } \pm 1, \pm 2, \pm \frac{1}{3}, \pm \frac{2}{3}}$

Example: Find the rational zeros of $f(x)=3x^4+4x^3+7x^2+8x+2.$

Use calculator to plug each potential rational zero from list above into the function.

$f(1)=24$ $\quad f(1/3)=5.6296$
$f(-1)=0$ $\quad f(-1/3)=0$
$f(2)=126$ $\quad f(2/3)=12.222$
$f(-2)=30$ $\quad f(-2/3)=-0.815$

$\boxed{\text{Rational zeros: } -1 \;\&\; -\frac{1}{3}}$

Theorem: Every polynomial function with real coefficients can be uniquely factored over the real numbers into a product of linear factors and/or irreducible (prime) quadratic factors.

Finding the Zeros of a Polynomial Function:

1. Use the degree of the polynomial to determine the number of zeros.
2. Use the Rational Zeros Theorem to identify the possible rational zeros.
3. Use substitution, synthetic division, or long division to test each potential rational zero. (It's easiest to use the Factor Theorem – plug the potential zeros into the function and see which ones give you 0.)
4. Each time a zero is found, use synthetic division to rewrite the function in factored form. Repeat step 3 on the ***depressed function*** (the function that remains after factoring out a linear factor).
5. Once the depressed function is quadratic, you can factor it or use the quadratic formula to find the remaining zeros.

★ Don't forget to use factoring techniques you already know to help you find the zeros.

Example: Find all of the real zeros of $f(x) = 2x^4 - x^3 - 5x^2 + 2x + 2$. Use the zeros to factor f over the real numbers. Degree 4 → max of 4 real zeros

Factors of constant (2): $\pm1, \pm2$ ← top

Factors of leading coefficient (2): $\pm1, \pm2$ ← bottom

Potential rational zeros: $\pm1, \pm2, \pm\frac{1}{2}$ Test in calculator: $f(1)=0$ $f(-\frac{1}{2})=0$

$(2x^4-x^3-5x^2+2x+2) \div (x-1)$

$$\begin{array}{r|rrrrr} 1 & 2 & -1 & -5 & 2 & 2 \\ & \downarrow & 2 & 1 & -4 & -2 \\ \hline -\frac{1}{2} & 2 & 1 & -4 & -2 & 0 \\ & \downarrow & -1 & 0 & 2 & \\ \hline & 2 & 0 & -4 & 0 & \end{array}$$

$(2x^3+x^2-4x-2) \div (x+\frac{1}{2})$

$f(x) = (x-1)(2x^3+x^2-4x-2)$

$f(x) = (x-1)(x+\frac{1}{2})(2x^2-4)$

$f(x) = 2(x-1)(x+\frac{1}{2})(x^2-2)$

$x^2-2=0$

$x^2=2$

$x=\pm\sqrt{2}$

$\boxed{\text{Zeros: } 1, -\frac{1}{2}, \sqrt{2}, -\sqrt{2}}$

$\boxed{\text{Factored form: } f(x) = 2(x-1)(x+\frac{1}{2})(x-\sqrt{2})(x+\sqrt{2})}$

Example: Find all real solutions of the equation $2x^3 - 3x^2 - 3x - 5 = 0$.

Same as asking "Find all the real zeros of $f(x) = 2x^3-3x^2-3x-5$"

Degree 3 → Max 3 solutions

Factors of constant (-5): $\pm1, \pm5$

Factors of leading coefficient (2): $\pm1, \pm2$

Potential rational zeros: $\pm1, \pm5, \pm\frac{1}{2}, \pm\frac{5}{2}$ Test in calculator: $f(\frac{5}{2})=0$

$$\begin{array}{r|rrrr} \frac{5}{2} & 2 & -3 & -3 & -5 \\ & \downarrow & 5 & 5 & 5 \\ \hline & 2 & 2 & 2 & 0 \end{array}$$

$(x-\frac{5}{2})(2x^2+2x+2)=0$

$2(x-\frac{5}{2})(x^2+x+1)=0$

$x=\frac{5}{2}$ or $x = \dfrac{-1\pm\sqrt{1^2-4(1)(1)}}{2(1)}$

$\boxed{x=\frac{5}{2} \text{ is the only real solution}}$

Complex Zero: A complex zero is in the form $a+bi$ or $a-bi$. $x = \dfrac{-1\pm\sqrt{-3}}{2} = \dfrac{-1\pm i\sqrt{3}}{2}$ ← imaginary

Fundamental Theorem of Algebra: Every complex polynomial function $f(x)$ of degree $n \geq 1$ can be factored into n linear factors (not necessarily distinct) of the form $f(x) = a_n(x-r_1)(x-r_2)...(x-r_n)$, where $a_n, r_1, r_2, ..., r_n$ are complex numbers. That is, every complex polynomial function of degree $n \geq 1$ has exactly n complex zeros, some of which may repeat.

Conjugate Pairs Theorem: For any complex polynomial function, if $r = a + bi$ is a zero of f, the complex conjugate $\overline{r} = a - bi$ is also a zero of f. ★ Imaginary solutions always come in pairs.

Corollary: A polynomial f of odd degree with real coefficients has at least one real zero.

Example: A polynomial f of degree 5 has the zeros 2, 3*i*, and 4 + *i*. Find the other two zeros.

$-3i$ & $4-i$

Example: Find a polynomial f of degree 4 that has the zeros 5, –3, and –2 + *i*. other zero: $-2-i$

zeros: $x=5$ $\quad x=-3$ $\quad x=-2+i$ $\quad x=-2-i$

factors: $x-5=0$ $\quad x+3=0$ $\quad x+2-i=0$ $\quad x+2+i=0$

$f(x)=(x-5)(x+3)(x+2-i)(x+2+i)$

$f(x)=(x^2-2x-15)(x^2+4x+5)$

$(x-5)(x+3)$
$= x^2+3x-5x-15$
$= x^2-2x-15$

	x	$+2$	$-i$
x	x^2	$+2x$	$\cancel{-ix}$
$+2$	$+2x$	$+4$	$\cancel{-2i}$
$+i$	$\cancel{+ix}$	$\cancel{+2i}$	$-i^2 = +1$

x^2+4x+5

	x^2	$-2x$	-15
x^2	x^4	$-2x^3$	$-15x^2$
$+4x$	$+4x^3$	$-8x^2$	$-60x$
$+5$	$+5x^2$	$-10x$	-75

$\boxed{f(x)=x^4+2x^3-18x^2-70x-75}$

Example: Find a polynomial f of degree 5 with the following zeros: 1, *i*, 4 – 2*i*. other zeros: $-i$, $4+2i$

zeros: $x=1$ $\quad x=i$ $\quad x=-i$ $\quad x=4-2i$ $\quad x=4+2i$

factors: $x-1=0$ $\quad x-i=0$ $\quad x+i=0$ $\quad x-4+2i=0$ $\quad x-4-2i=0$

$f(x)=(x-1)(x-i)(x+i)(x-4+2i)(x-4-2i)$

$f(x)=(x-1)(x^2+1)(x^2-8x+20)$

$f(x)=(x^3-x^2+x-1)(x^2-8x+20)$

$(x-i)(x+i)$
$= x^2\cancel{+ix}\cancel{-ix}-i^2$
$= x^2+1$

$(x-1)(x^2+1)$
$= x^3+x-x^2-1$
$= x^3-x^2+x-1$

	x	-4	$+2i$
x	x^2	$-4x$	$\cancel{+2ix}$
-4	$-4x$	$+16$	$\cancel{-8i}$
$-2i$	$\cancel{-2ix}$	$\cancel{+8i}$	$-4i^2 = +4$

$x^2-8x+20$

	x^3	$-x^2$	$+x$	-1
x^2	x^5	$-x^4$	$+x^3$	$-x^2$
$-8x$	$-8x^4$	$+8x^3$	$-8x^2$	$+8x$
$+20$	$+20x^3$	$-20x^2$	$+20x$	-20

$\boxed{f(x)=x^5-9x^4+29x^3-29x^2+28x-20}$

Example: Find the complex zeros of $f(x) = 2x^4 + x^3 - 35x^2 - 113x + 65$. Write f in factored form.

Degree $4 \rightarrow 4$ complex zeros

Factors of constant (65): $\pm 1, \pm 65, \pm 5, \pm 13$

Factors of leading coefficient (2): $\pm 1, \pm 2$

Potential rational zeros:
$\pm 1, \pm 65, \pm 5, \pm 13, \pm \frac{1}{2}, \pm \frac{65}{2}, \pm \frac{5}{2}, \pm \frac{13}{2}$

$f(5) = 0 \quad f\left(\frac{1}{2}\right) = 0$

$$\begin{array}{r|rrrrr} 5 & 2 & 1 & -35 & -113 & 65 \\ & \downarrow & 10 & 55 & 100 & -65 \\ \hline \frac{1}{2} & 2 & 11 & 20 & -13 & \vert\ 0 \\ & \downarrow & 1 & 6 & 13 & \\ \hline & 2 & 12 & 26 & \vert\ 0 & \end{array}$$

$f(x) = (x-5)\left(x - \frac{1}{2}\right)(2x^2 + 12x + 26)$

$f(x) = 2(x-5)\left(x - \frac{1}{2}\right)(x^2 + 6x + 13)$

$$x = \frac{-6 \pm \sqrt{6^2 - 4(1)(13)}}{2(1)} = \frac{-6 \pm \sqrt{-16}}{2} = \frac{-6 \pm 4i}{2} = -3 \pm 2i$$

$\boxed{\text{zeros: } 5, \frac{1}{2}, -3+2i, -3-2i}$

$\boxed{\text{factored form: } f(x) = 2(x-5)\left(x-\frac{1}{2}\right)(x+3-2i)(x+3+2i)}$

Example: Find the complex zeros of $f(x) = x^4 + 3x^3 - 19x^2 + 27x - 252$. Write f in factored form.

Degree $4 \rightarrow 4$ complex zeros

Factors of constant (252): $\pm 1, \pm 252, \pm 2, \pm 126, \pm 3, \pm 84, \pm 4, \pm 63, \pm 6, \pm 42, \pm 7, \pm 36, \pm 9, \pm 28, \pm 12, \pm 21, \pm 14, \pm 18$

Factors of leading coefficient (1): ± 1

Potential rational zeros

$f(4) = 0 \quad f(-7) = 0$

$$\begin{array}{r|rrrrr} 4 & 1 & 3 & -19 & 27 & -252 \\ & \downarrow & 4 & 28 & 36 & 252 \\ \hline -7 & 1 & 7 & 9 & 63 & \vert\ 0 \\ & \downarrow & -7 & 0 & -63 & \\ \hline & 1 & 0 & 9 & \vert\ 0 & \end{array}$$

$f(x) = (x-4)(x+7)(x^2+9)$

$x^2 + 9 = 0$

$x^2 = -9$

$x = \pm\sqrt{-9}$

$x = \pm 3i$

$\boxed{\text{zeros: } 4, -7, 3i, -3i}$

$\boxed{\text{factored form: } f(x) = (x-4)(x+7)(x-3i)(x+3i)}$

Example: One of the zeros of $f(x) = x^4 - 2x^3 - 3x^2 + 10x - 10$ is $1+i$. Find the remaining zeros.

$1-i$ is also a zero.

None of the potential rational zeros work.

$(x-1-i)$ & $(x-1+i)$ are factors,

so $(x-1-i)(x-1+i) = x^2 - 2x + 2$ is also a factor.

	x	-1	$-i$
x	x^2	$-x$	$\cancel{-ix}$
-1	$-x$	$+1$	$\cancel{+i}$
$+i$	$\cancel{+ix}$	$\cancel{-i}$	$-i^2 = +1$

$x^2 - 2x + 2$

$x^4 - 2x^3 - 3x^2 + 10x - 10 = (x^2 - 2x + 2)(\ ?\ )$

$$\begin{array}{r} x^2 - 5 \qquad\qquad\qquad \\ x^2 - 2x + 2 \overline{\big)\, x^4 - 2x^3 - 3x^2 + 10x - 10} \\ -(x^4 - 2x^3 + 2x^2) \qquad\qquad \\ \hline -5x^2 + 10x - 10 \\ -(-5x^2 + 10x - 10) \\ \hline 0 \end{array}$$

Remaining factor is $x^2 - 5$

$x^2 - 5 = 0$

$x^2 = 5$

$x = \pm\sqrt{5}$

$\boxed{\text{Remaining zeros: } 1-i, \sqrt{5}, -\sqrt{5}}$