Rates of Change and Tangent Lines

In 2.1 we computed average rates of change. We looked at $s=16t^2$ (s=position, t=time) and average rate of change for 1 second to 2 seconds. Computing average rates of change (i.e. slopes) is important because we see them all the time, average speed, average population, growth, annual percentage rules, monthly rainfall, price of gas…etc. All are average rates of change.

Example: $f(x)=x^3-x$ Find the average rate of change over [1,3]
Because f(1) = 0 and f(3) = 24 the average rate of change is

$$\frac{f(3)-f(1)}{3-1}=\frac{24-0}{3-1}=12$$

$$or\ \frac{(f\ at\ end\ of\ \text{interval})-(f\ at\ start\ of\ \text{interval})}{(end\ of\ \text{interval})-(start\ of\ \text{interval})}$$

$$or\ \frac{\Delta y}{\Delta x}\ or\ \frac{dy}{dx}$$

Scientists are sometimes interested with how a population is changing with respect to time…i.e. "What is the rate of change?" (See example 1 and 2)

A "secant" line is a graphical representation of the rate of change.

Example: Find the average rate of change from 23 to 45.
The slope of the secant line represents the average rate of change.
Examine the graph – was the rate of change 9 flies/day every day?
Where was it more?
Where was it less?

Maybe we would like to know the rate of change at exactly day 23.

Look at Figure 2.28 and make interval smaller and smaller, slope of secant gets steeper.
As Q approaches P the secant approaches the tangent line at P. Using A &B as approximates of the points on the tangent line: B (35, 350) and A (15,0) then $\frac{350-0}{35-15}=17.5$

When computing the rate at 23 we talked about Q approaching P and the secant approaching the tangent. This vocabulary suggests a limit will be involved.

With this example it suggests the rate of change at a point should be defined as the slope of the tangent to the curve at that point. This suggests slope at one point $m=\frac{y_2-y_1}{x_2-x_1}$ The denominator is zero so how can we do this? Luckily Pierre de Fermat figured this out.

3 Steps to Finding Tangent

Step 1: We start with what we can calculate, namely, the slope of a secant through P and a point Q nearby on the curve.
Step 2: We find the limiting value of the secant slope (if it exists) as Q approaches P along the curve.
Step 3: We define the slope of the curve at P to be this number and define the tangent to the curve at P to be the line through P with this slope.

Example: f(x) = x^2, find the tangent to the curve when x = 2

Step 1: use P (2, f(2)) and Q (2+h, f(2+h))

Step 2:

$$m=\frac{(2+h)^2-(2)^2}{2+h-2}\rightarrow\lim_{h\to 0}\frac{(2+h)^2-(2)^2}{2+h-2}=\lim_{h\to 0}\frac{h^2+4h+4-4}{h}=\lim_{h\to 0}\frac{h^2+4h}{h}=\lim_{h\to 0}h+4=4$$

Step 3: the slope of the tangent to the curve at f(x) = x^2 at x = 2 (2,4) is 4.

For this problem, we did $\lim_{h\to 0}\frac{f(2+h)-f(2)}{2+h-2}\rightarrow\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$

What is to stop us from doing the same thing at any point x?

Definition: Slope of a Curve at a point

The slope of the curve y=f(x) at the point P(a,f(a)) is the number $m=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ provided the limit exists.
The tangent line to the curve at P is the line through P with this slope.

Look at example #4

Another example: f(x) = x^2+4x Graph the function.

Find the slope of the curve at x=1

$$\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0}\frac{(1+h)^2+4(1+h)-f(1)}{h}=\lim_{h\to 0}\frac{1^2+2h+h^2+4(1+h)-(1^2+4(1))}{h}$$

$$=\lim_{h\to 0}\frac{1^2+2h+h^2+4+4h-1^2-4}{h}$$

$$=\lim_{h\to 0}\frac{h^2+2h+4h}{h}=\lim_{h\to 0}h+6=6$$

Find the slope of the curve at x=a

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{h\to 0}\frac{(a+h)^2+4(a+h)-f(a)}{h}=\lim_{h\to 0}\frac{a^2+2ah+h^2+4(a+h)-(a^2+4a)}{h}$$

$$=\frac{a^2+2ah+h^2+4a+4h-a^2-4a}{h}$$

$$=\frac{h^2+2ah+4h}{h}=h+2a+4 \quad = \quad 2a+4$$

Where does the slope equal 0? Answer: (As the tangent line approaches x=-2)

What happens to the tangent to the curve for different values of a?
Answer: (For a < -2 slope of tangent is negative; for a>-2 the slope of the tangent is positive.)

Remember $\frac{f(a+h)-f(a)}{h}$ is called the **difference quotient** of the function at *a*.

The **difference quotient has two main interpretations**. Graphically it is the **slope of the tangent line at x=a**. Analytically we interpret this to be **the rate of change of the function with respect to the change in x.**

Recall, change in function values f(x) = y was Δy; change is x's was Δx and $\frac{\Delta y}{\Delta x}$ is change in y at x.

The $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ is one of the two most important concepts in calculus.

Related topic – **Normal line**. Normal means **perpendicular**

Example: $f(x) = x^2+4x$
Find the equation of the tangent and the normal line to the curve at x=1.
The equation of the tangent line to the curve at x=1. The slope was 6. The point at (1, f(1)) = (1,5) so …

$$y-5 = 6(x-1)$$
$$y = 6x-6+5$$
$$y = 6x-1$$

We found the slope to be 6 at x=1; so the slope of the normal is -1/6.
The normal to the curve at (1, f(1)) = (1,5) is the line through (1,5) with slope = -1/6.

$$y-5 = \frac{-1}{6}(x-1)$$
$$y = \frac{-1}{6}x+\frac{1}{6}+5$$
$$y = \frac{-1}{6}x+\frac{1}{6}+\frac{30}{6}$$
$$y = \frac{-1}{6}x+\frac{31}{6}$$