Polynomial Functions and Graphs

A ***polynomial function*** is a function in the form $f(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0$ where $a_n,\ a_{n-1},\ldots,\ a_1,\ a_0$ are real numbers (coefficients) and n is a nonnegative integer. The domain of a polynomial function is the set of all real numbers. The ***degree*** of a polynomial is the largest power of x that appears. The zero polynomial $f(x)=0$ is not assigned a degree.

No variables under roots (exponents aren't integers)
No variables in denominator (negative exponent)

Example: Determine which of the following are polynomial functions. For those that are, state the degree; for those that are not, tell why not.

a) $f(x)=5+2x^2-8x^3$

polynomial, degree 3

b) $f(x)=4+3\sqrt{x}$ $=4+3x^{\frac{1}{2}}$

not a polynomial, non-integer exponent

c) $f(x)=-2x^3(x-1)^2$

polynomial, degree 5
(if multiplied out, leading term would be $-2x^5$)

d) $f(x)=0$

polynomial, not assigned a degree
$0x^1$? $0x^{27}$? $0x^6$?

e) $f(x)=\dfrac{x^2-1}{x+4}$

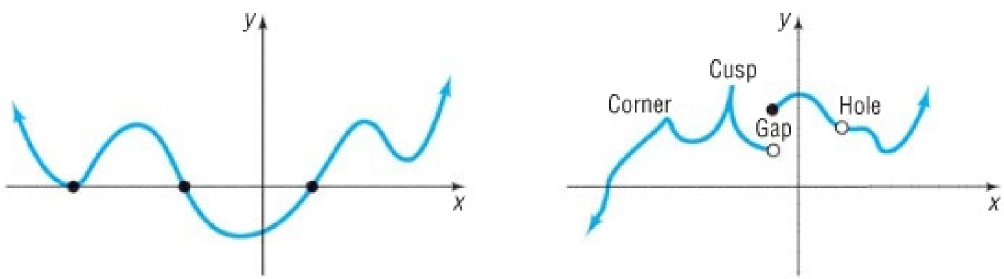
$(x^2-1)(x+4)^{-1}$

not a polynomial, negative exponent

f) $f(x)=9$

polynomial, degree 0
$9x^0$

A polynomial function is smooth and continuous. It does not contain corners, cusps, gaps, or holes.

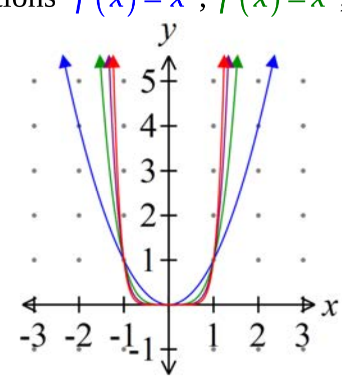


(a) Graph of a polynomial function: smooth, continuous

(b) Cannot be the graph of a polynomial function

A ***power function of degree n*** is a monomial of the form $f(x)=ax^n$, where a is a real number, $a\neq 0$, and n is an integer greater than 0.

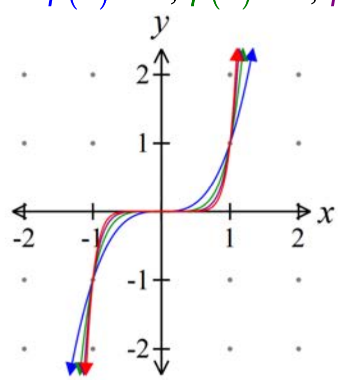
Example: Compare the even power functions $f(x)=x^2$, $f(x)=x^4$, $f(x)=x^6$, and $f(x)=x^8$.



Properties of Power Functions $f(x) = x^n$, n Is an Even Integer

1. f is an even function, so its graph is symmetric with respect to the *y*-axis.
2. The domain is the set of all real numbers. The range is the set of nonnegative real numbers.
3. The graph always contain the points (–1, 1), (0, 0), and (1, 1).
4. As the exponent *n* increases in magnitude, the graph becomes more vertical when $x < -1$ or $x > 1$; but for *x* near the origin, the graph tends to flatten out and lie closer to the *x*-axis.

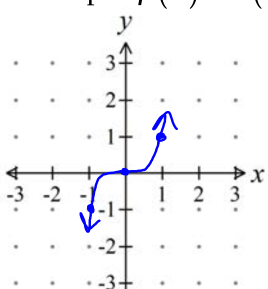
Example: Compare the odd power functions $f(x) = x^3$, $f(x) = x^5$, $f(x) = x^7$, and $f(x) = x^9$.



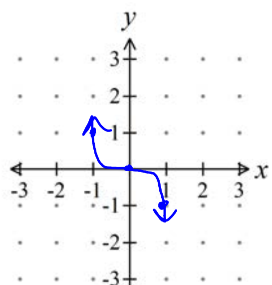
Properties of Power Functions $f(x) = x^n$, n Is an Odd Integer

1. f is an odd function, so its graph is symmetric with respect to the origin.
2. The domain and the range are the set of all real numbers.
3. The graph always contain the points (–1, –1), (0, 0), and (1, 1).
4. As the exponent *n* increases in magnitude, the graph becomes more vertical when $x < -1$ or $x > 1$; but for *x* near the origin, the graph tends to flatten out and lie closer to the *x*-axis.

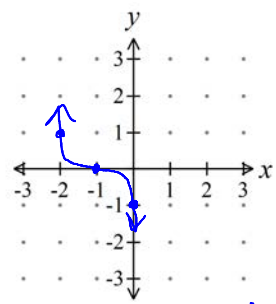
Example: Graph $f(x) = -(x+1)^5$ using transformations.



$y = x^5$

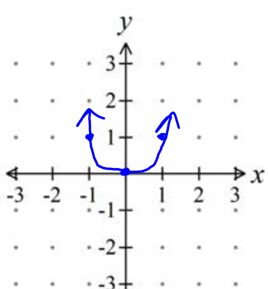


$y = -x^5$

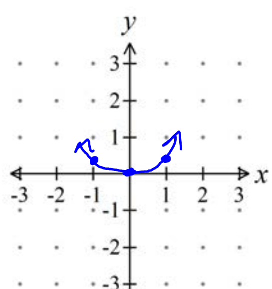


$y = -(x+1)^5$

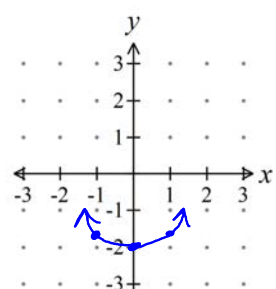
Example: Graph $f(x) = \frac{1}{3}x^4 - 2$ using transformations.



$y = x^4$



$y = \frac{1}{3}x^4$



$y = \frac{1}{3}x^4 - 2$

If f is a function and r is a real number for which $f(r)=0$, then r is called a ***real zero*** of f. The real zeros of a polynomial function are the x-intercepts of its graph, and they are found by solving the equation $f(x)=0$.

The following statements are equivalent:

1. r is a solution of $f(x)=0$.
2. r is a real zero of a polynomial function f.
3. r is an x-intercept of the graph of f.
4. $x-r$ is a factor of f.

Example:

a) Find a polynomial of degree 3 whose zeros are –2, 4, and 5.

b) Use a graphing calculator to graph the polynomial and verify the results of part a).

$f(x)=(x+2)(x-4)(x-5)$

$f(x)=(x+2)(x^2-5x-4x+20)$

$f(x)=(x+2)(x^2-9x+20)$

$f(x)= x^3-9x^2+20x$
$\quad +2x^2-18x+40$

$\boxed{f(x)=x^3-7x^2+2x+40}$

If the same factor $x-r$ occurs more than once, r is called a ***repeated,*** or ***multiple zero* of f.** More precisely, if $(x-r)^m$ is a factor of a polynomial f and $(x-r)^{m+1}$ is not a factor of f, then r is called a ***zero of multiplicity m* of f.**

Example: Identify the zeros and their multiplicities for the polynomial $f(x)=3(x-4)(x+1)^3(x-2)^2$.

$0=3(x-4)(x+1)^3(x-2)^2$

zero:	4	-1	2
multiplicity:	1	3	2

$x-4=0$
$x=4$

$(x+1)^3=0$
$x=-1$

$(x-2)^2=0$
$x=2$

Multiplicity Rules:

If r is a zero of even multiplicity:

1. The sign of $f(x)$ does not change from one side to the other side of r.
2. The graph of f **touches** the x-axis at r.

If r is a zero of odd multiplicity:

1. the sign of $f(x)$ changes from one side to the other of r.
2. The graph of f **crosses** the x-axis at r.

Turning Points: The points at which a graph changes direction (the local maxima and local minima) are called turning points. If f is a polynomial function of degree n, then f has at most $n-1$ turning points. If the graph of a polynomial f has $n-1$ turning points, the degree of f is at least n.

End Behavior: The behavior of the graph of a function for large values of x, either positive or negative, is referred to as its end behavior. For large values of x, either positive or negative, the graph of the polynomial function $f(x)=a_nx^n+a_{n-1}x^{n-1}+...+a_1x+a_0$ resembles the graph of the power function $y=a_nx^n$. (The ends of the graph are pointing the same direction as the ends of the graph of the leading term of the polynomial).

Example: Graph the polynomial $f(x) = x^2(x+3)$.

a) Find the *x*-intercepts and *y*-intercept of the graph of f.
b) Determine whether the graph crosses or touches the *x*-axis at each *x*-intercept.
c) End behavior: Find the power function that the graph of f resembles for large values of $|x|$.
d) Determine the maximum number of turning points on the graph of f.
c) Locate other points on the graph and connect all the points plotted with a smooth continuous curve.

x-ints/zeros: $0 = x^2(x+3)$

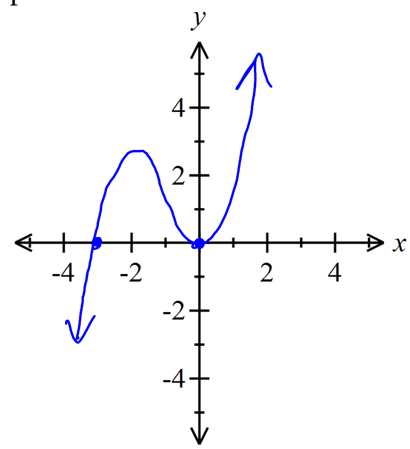
$x^2 = 0$ $\quad$ $x+3=0$

$x = 0$ $\quad$ $x = -3$

mult 2 $\quad$ mult 1

touches $\quad$ crosses

y-int: $f(0) = 0^2(0+3) = 0$



End behavior: $y = x^3$ $\downarrow$ $\uparrow$

Max turning pts = degree − 1

$= 3 - 1 = 2$

Example: Graph the polynomial $f(x) = (x+1)^2(x-3)(x-1)$.

a) Find the *x*-intercepts and *y*-intercept of the graph of f.
b) Determine whether the graph crosses or touches the *x*-axis at each *x*-intercept.
c) End behavior: Find the power function that the graph of f resembles for large values of $|x|$.
d) Determine the maximum number of turning points on the graph of f.
c) Locate other points on the graph and connect all the points plotted with a smooth continuous curve.

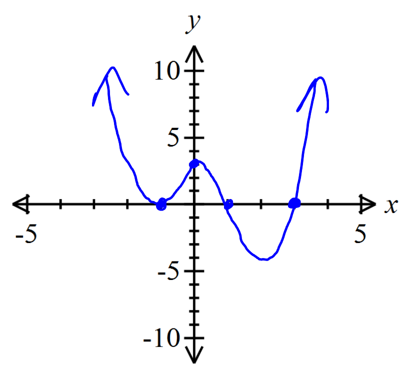
x-ints/zeros: $0 = (x+1)^2(x-3)(x-1)$

$(x+1)^2 = 0$ $\quad$ $x-3=0$ $\quad$ $x-1=0$

$x = -1$ $\quad$ $x = 3$ $\quad$ $x = 1$

mult 2 $\quad$ mult 1 $\quad$ mult 1

touches $\quad$ crosses $\quad$ crosses



y-int: $f(0) = (0+1)^2(0-3)(0-1)$

$= 1^2(-3)(-1) = 3$

End behavior: $y = x^4$ $\uparrow$ $\uparrow$

Max. turning pts. $= 4 - 1 = 3$

Example: Graph the polynomial $f(x) = -\frac{1}{2}x\left(x^2+1\right)(x-2)^2$.

a) Find the *x*-intercepts and *y*-intercept of the graph of f.

b) Determine whether the graph crosses or touches the *x*-axis at each *x*-intercept.

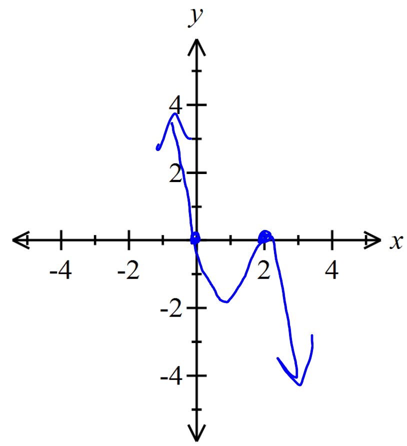
c) End behavior: Find the power function that the graph of f resembles for large values of $|x|$.

d) Determine the maximum number of turning points on the graph of f.

c) Locate other points on the graph and connect all the points plotted with a smooth continuous curve.

x-ints: $0 = -\frac{1}{2}x(x^2+1)(x-2)^2$

$-\frac{1}{2}x = 0$	$x^2+1=0$	$(x-2)^2=0$
$x=0$	$x^2=-1$	$x=2$
mult. 1	$x = \pm\sqrt{-1}$	mult. 2
crosses	no x-ints from this factor	touches



y-int: $f(0) = -\frac{1}{2}(0)(0^2+1)(0-2)^2 = 0$

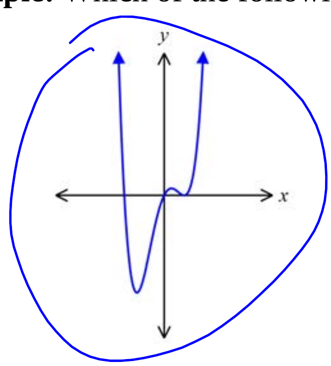
End behavior: $y = -\frac{1}{2}x^5$ ↖ ↘

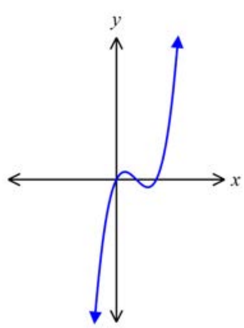
Max turning pts: $5-1=4$

Example: Which of the following could be the graph of $f(x) = x^4 - 3x^2 + 2x$? ← End behavior: $y = x^4$ ↑↑

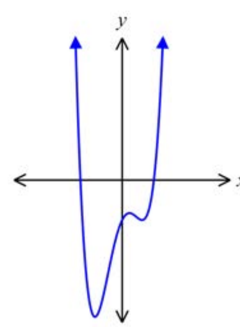
y-int: $f(0)=0$

Max turning pts $= 4-1=3$





No. Wrong end behavior



No. Wrong y-int