Limits Involving Infinity

Last time we talked about the general idea of what a limit is and found some nice properties that made a complicated idea easy to figure. Each time we asked what happens as $x \to c$ or $\lim_{x\to c} f(x)$. Now we look at limits as x goes to ∞ or -∞ or $\lim_{x\to\infty} f(x)$. ∞ is not a number we can plug in. We need to see what happens to f (x) as x moves further to the right toward ∞. The same is true for -∞, we can have $\lim_{x\to -\infty} f(x)$. We figure it the same way.

Example: Think of $f(x)=\frac{1}{x}$ the inverse (reciprocal) function. As $x\to\infty$ f(x)=0 so

$\lim_{x\to\infty}\frac{1}{x}=0 \quad \lim_{x\to-\infty}\frac{1}{x}=0$ When a graph "straightens out" it is said to have asymptotic behavior. The graph approaches a line. In this example the line it approaches is horizontal. It has a horizontal asymptote. (y = 0).

In general if $\lim_{x\to\infty} f(x)=b \quad or \quad \lim_{x\to-\infty} f(x)=b$ then y=b is a horizontal asymptote.

Example: $y=\frac{1}{x}-1 \qquad \lim_{x\to\infty}\frac{1}{x}-1=-1$ y = -1 is the horizontal asymptote

What a graph starts to look like is also called the end behavior. Calculators can be helpful although we shouldn't rely on it. Approach problems graphically, numerically, and analytically

Example: $f(x)=\frac{x}{\sqrt{x^2+1}}$

Example: $f(x)=\frac{\sin(x)}{x}$

Recall Sandwich theorem:

$$-1\le \sin(x)\le 1 \quad so \quad \frac{-1}{x}\le\frac{\sin(x)}{x}\le\frac{1}{x}$$

$$\lim_{x\to\infty} \quad \lim_{x\to\infty} \quad \lim_{x\to\infty}$$

$$0\le \qquad \le 0 \qquad \frac{\sin x}{x} is \; even$$

Just as with limits as $x\to c$, we have nice properties of limits as $x\to\pm\infty$. Theorem 5 gives the properties of these limits. They are Sum, difference, product, constant multiple, quotient and power rules.

Example:

$$f(x)=\frac{1}{x^2} \quad find \; \lim_{x\to 2} \; simply \; plug \; in,$$

$$but \; find \; \lim_{x\to 0}\frac{1}{x^2} \; you \; can't \; plug \; in \ldots graph$$

Does f(x) have a limit as $x\to 0$? Remember $\lim_{x\to c^+} must \; equal \; \lim_{x\to c^-}$ In this case, yes it does.

Example: $h(x) = \frac{1}{x}$ The limits are not equal, so there is no limit.

$\lim_{x \to 0^-} h(x) = -\infty \quad \lim_{x \to 0^+} h(x) = +\infty$

What about asymptotes? x= 0

$If \lim_{x \to a^+} f(x) = \pm\infty \; or \lim_{x \to a^-} f(x) = \pm\infty$ then the x=*a* is a vertical asymptote

What are some functions that have vertical asymptotes? Rational functions & Trig functions

Example: $f(x) = \frac{x}{x^2 - 4} \quad or \quad g(x) = \tan(x) = \frac{\sin x}{\cos x}$

You might think that asymptotes occur whenever the denominator is zero. Not always true!

Consider $f(x) = \frac{\sin(x)}{x}$

Earlier we talked about end behavior. What happens for "very large values" of x?

Definition: End behavior Model:

The function *g* is

(a) a right end behavior model for *f* if and only if $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1$

(b) a left end behavior model for *f* if and only if $\lim_{x \to -\infty} \frac{f(x)}{g(x)} = 1$

The right end and left end behavior do not need to be the same.

Example: $f(x) = 2x^3 - 3x + 4$

Which terms really make a difference as x gets really large? Only $2x^3$ so $g(x) = 2x^3$ is an end behavior model. Analytically $\lim_{x \to \infty} \frac{2x^3 - 3x + 4}{2x^3} = \frac{2x^3}{2x^3} - \frac{3x}{2x^3} + \frac{4}{2x^3} = 1$

Find a simple basic function EBM for $f(x) = e^{-x} - 2x$

Use end behavior to find asymptotes:

$f(x) = \frac{2x^5 + x^4 - x^2 + 1}{3x^2 - 5x + 7} \qquad f(x) = \frac{2x^3 - x^2 + x - 1}{5x^3 + x^2 + x - 5} \qquad f(x) = \frac{x - 2}{2x^2 + 3x - 5}$

There is an interesting truth that can be used to find difficult limits. If $x \to \infty$ what happens to $\frac{1}{x}$? It approaches 0. We can use this to rewrite limits.

Example:

$\lim_{x \to \infty} xe^x = ?$

$\lim_{x \to \infty} f(x) = \lim_{x \to 0} f\left(\frac{1}{x}\right) \quad so \quad \lim_{x \to 0} \frac{e^{\frac{1}{x}}}{x}$