2.1notes calculus

Rates of Change and Limits

Today we will talk about a key idea in calculus – an idea that separates calculus from "lower" kinds of math –
Example: Student walking ½ the distance to the wall. How close will the student get? Will he/she ever actually get to the wall? What happens to the distance?

In calculus we discuss ideas like this. In comparison in algebra or pre-calculus we might have said how many steps need to be taken to get to the wall? How long will it take? Etc.

Another example:
Sketch the graph of $y = x^3 - 4x + 2$
What does the graph look like at x = -1? Zoom in..
How long could this go on?

In calculus many times we will deal with real life data, position, velocity, acceleration.

Take a look at example 1 and recall that $velocity = \frac{distance}{time}$ and $average\,velocity = \frac{\Delta distance}{\Delta time}$

$y = 16t^2$ *feet* $y = distance(position)$ $t = time$

$$average\,velocity = \frac{\Delta y}{\Delta x} = \frac{end\ position - start\ position}{end\ time - start\ time} = \frac{16(2)^2 - 16(0)}{2-0} = 32\frac{ft}{\sec}$$

What does time mean? Was the rock always going 32 ft/sec? Did it go faster? Slower?
What if we want to know exactly how fast it was going at t=2?
There are several ways to find this.

What if we find the average speed at a small interval around 2? How about a really small interval say t=2 and t=2+h where h is a little bit.

$$\frac{\triangle y}{\triangle x} = \frac{16(2+h)^2 - 16(2)^2}{2+h-2}$$
$$= \frac{16(2+h)^2 - 16(2)^2}{h}$$

This is called a difference quotient. "kind of like slope" In Example 2 we want speed at t = 2. Remember, the original problem is not at about 2. It is exactly at 2. We call this the instantaneous velocity at 2. The difference quotient can't be used to find velocity at 2 because h would need to be zero. So, here's the key. (trick)

Rearrange

$$\frac{16(4+4h+h^2)-64}{h}=\frac{64+64h+16h^2-64}{h}=\frac{64h+16h^2}{h}=64+16h \; if \; h\neq 0$$

$64 \, ft/\sec$

Let's look at the idea of what happens to the function as a approaches a certain value.

Limit example from trig: $f(x)=\frac{\sin x}{x}$

What happens as x gets close to zero?
What happens at x=0?

Look at the Graph table: What happens to the function as x approaches 0? The function approaches 1.

Definition: Limit
Let c and L be real numbers. The function f has limit L as x approaches c if, given any positive number ε, there is a positive number δ such that for all x,

$$0<|x-c|<\delta \Rightarrow |f(x)-L|<\varepsilon$$

$$We\,write \lim_{x\to c} f(x)=L$$

Tell me how close you want to get to an x value and I'll tell you how close that is to a y value. It is not important that we use ε and δ , but we will use limit notation.

$$\lim_{h\to 0}\frac{16(2+h)^2-16(2)^2}{h} \qquad \lim_{h\to 0}\frac{\sin(x)}{x} \qquad \lim_{x\to 2}\frac{x^2-4}{x-2}$$

Look at Figure 2.2
Limits have nothing to do with what is happening to f(x) at c, it has everything to do with what's happening to f(x) as x approaches c.

This is not all as confusing as it sounds, and each limit is not an aweful process to go through. There are nice properties of limits that make things easier once we have the basics in hand.

Theorem 1 page 61

Examples:

$$\lim_{x\to 3} x^2+2 \qquad sum$$

$$\lim_{x\to 2} \frac{x^3+x^2+2}{x^2-1} \qquad sum, difference, quotient$$

$$\lim_{x\to c} x^4+3x^3-2 \qquad sum \,\&\, difference$$

$$\lim_{x\to c} \frac{x^3+3}{x^2+2} \qquad sum \,\&\, quotient$$

Example: $\lim_{x\to b}\left(\frac{x^2+2}{x^4+1}\right)^{\frac{1}{2}}$

Theorem 2 (page 63) says if we are finding a limit involving polynomials, we can simply plug in the point we're interested in, provided we don't end up with a denominator of zero.

$$\lim_{x\to 0} \frac{\tan x}{x} \; break\, up \; \lim_{x\to 0} \tan x \cdot \frac{1}{x}$$

$$\lim_{x\to 0} \frac{\sin x}{\cos x}\cdot\frac{1}{x} \to \frac{\sin x}{x}\cdot\frac{1}{\cos x}$$

$$\lim_{x\to 0} \frac{\sin x}{x}\cdot \lim_{x\to 0}\frac{1}{\cos x}$$

$$previously: (1)\cdot(1) = 1$$

Graphical approach

$$\lim_{x\to 1} \frac{x^2+3}{x-1}$$

These ideas would be sufficient if we always had nice smooth graphs and a calculator but what about piecewise functions?

Consider

$$f(x) = \begin{Bmatrix} x^2 & x \le 2 \\ x & x > 2 \end{Bmatrix}$$

$$\lim_{x\to 2} f(x) =$$

From the right it is approaching what value?
From the left it is approaching what value?
Does it have a limit?

Theorem 3

One-sided and Two-sided limits

A function f(x) has a limit as x approaches c if and only if the right-hand and left-hand limits at c exist and are equal.

Right hand limit | *left hand* limit

$\lim\limits_{x \to c^+} f(x)$ | $\lim\limits_{x \to c^-} f(x)$

Limit means 2 sided limit.

Has limit if 1) both left and right exist 2) left limit =right limit

Finally

The Sandwich Theorem

If we can't find a limit directly, we might be able to find it indirectly by squeezing the function between 2 known functions that have the same limit as x→c (The point of interest)

Don't worry about how to find such a function.

$\lim\limits_{x \to 0} x \sin x$

Start with what you know.

$-1 \leq \sin x \leq 1$ *Now multiply all expressions by x.*

$-|x| \leq x \sin x \leq |x|$

$\lim\limits_{x \to 0} -|x| \leq \lim\limits_{x \to 0} x \sin x \leq \lim\limits_{x \to 0} |x|$

$0 \leq \lim\limits_{x \to 0} x \sin x \leq 0$