

Quadratic Functions and Their Properties

Quadratic Function: A function that can be written in the form $f(x) = ax^2 + bx + c$, where a , b , and c are real numbers and where $a \neq 0$. The domain of a quadratic function is the set of all real numbers.

Graphing a Quadratic Function Using Transformations

1. Begin with the parent function $f(x) = x^2$.
 2. Rewrite the function in vertex form $f(x) = a(x-h)^2 + k$ by completing the square.
 3. Transform with the following:
 - a : If a is positive, the graph opens up. The y-coordinate of the vertex is a minimum value.
If a is negative, the graph opens down. The y-coordinate of the vertex is a maximum value.
 - If $|a| > 1$, the graph is narrower than the graph of $f(x) = x^2$. (vertical stretch)
 - If $|a| < 1$, the graph is wider than the graph of $f(x) = x^2$. (vertical compression)
 - h : h is the *opposite* of the number with x . It controls the horizontal shift (left and right).
 - k : k controls the vertical shift (up and down).
- Vertex: (h, k) Axis of Symmetry: $x = h$

Completing the Square: Figuring out what constant to add to a binomial of the form $x^2 + bx$ to make it into a perfect square trinomial, then writing the result in factored form.

Example: Add the proper constant to the binomial to make it into a perfect square trinomial. Then factor the trinomial.

Completing the Square for the Binomial $x^2 + bx$

1. Divide the coefficient of the x -term by 2. (Find $\frac{b}{2}$).
2. Square the answer from step 1. (Find $(\frac{b}{2})^2$).
3. Add the result of step 2 to the binomial.
4. Rewrite as a perfect square: $(x + \frac{b}{2})^2$.

| | |
|------------------------------------------------------------|--------------------------------------------------------------------------------------------------|
| $x^2 + 12x + \underline{\quad}$ | $x^2 - 7x + \underline{\quad}$ |
| $\frac{12}{2} = 6$ $6^2 = 36$ $x^2 + 12x + 36$ $(x + 6)^2$ | $-\frac{7}{2}$ $(-\frac{7}{2})^2 = \frac{49}{4}$ $x^2 - 7x + \frac{49}{4}$ $(x - \frac{7}{2})^2$ |

Examples: Add the proper constant to each binomial to make it into a perfect square trinomial. Then factor the trinomial.

a) $x^2 + 16x + \underline{64} = (x + \underline{8})^2$ b) $x^2 - 3x + \underline{\frac{9}{4}} = (x - \underline{\frac{3}{2}})^2$ c) $x^2 + \frac{4}{3}x + \underline{\frac{4}{9}} = (x + \underline{\frac{2}{3}})^2$

$$\frac{16}{2} = 8$$

$$8^2 = 64$$

$$-\frac{3}{2}$$

$$(-\frac{3}{2})^2 = \frac{9}{4}$$

$$\frac{4}{3} \div 2 = \frac{2}{3}$$

$$(\frac{2}{3})^2 = \frac{4}{9}$$

Writing $f(x) = ax^2 + bx + c$ in Vertex Form

1. Group ax^2 and bx together in parentheses.
2. If $a \neq 1$, factor out a from $ax^2 + bx$. Include a negative if the quadratic term is negative.
3. Complete the square (divide b by 2 and square the result). Add the answer inside the parentheses. Keep the equation balanced by adding or subtracting outside the parentheses. (You are adding 0 to one side of the equation.) Remember to take into account that anything you add inside the parentheses is being multiplied by anything that was factored out!
4. Write the expression inside the parentheses as a perfect square.

Examples: Write each equation in vertex form by completing the square. Find the vertex, axis of symmetry, maximum or minimum value, and draw the graph of the function.

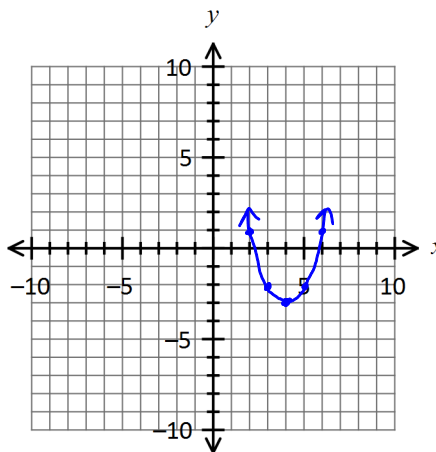
a) $f(x) = (x^2 - 8x) + 13$

$f(x) = (x^2 - 8x + 16) + 13 - 16$
added zero

$-\frac{b}{a} = -4$
 $(-4)^2 = 16$

$f(x) = (x - 4)^2 - 3$

vertex: $(4, -3)$
 axis of symmetry: $x = 4$
 minimum value = -3



b) $y = (-x^2 + 4x)$

Factor out the negative:

$y = -(x^2 - 4x)$

complete the square for $x^2 - 4x$

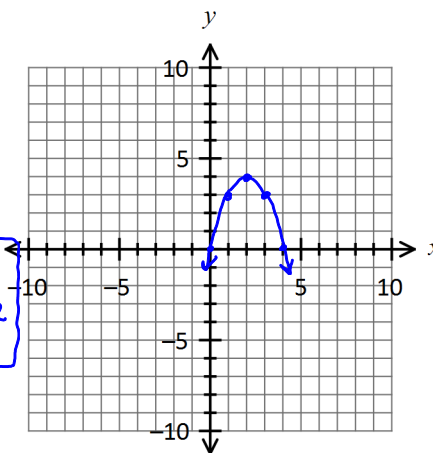
$y = -(x^2 - 4x + 4) + 4$

the negative applies to anything you put inside the parentheses, so you actually subtracted 4! Add 4 outside to have the net effect of adding zero.

$-\frac{4}{2} = -2$
 $(-2)^2 = 4$

$y = -(x - 2)^2 + 4$

vertex: $(2, 4)$
 axis of symmetry: $x = 2$
 maximum value = 4



c) $f(x) = (3x^2 + 6x) - 7$

Factor out 3:

$f(x) = 3(x^2 + 2x) - 7$

$\frac{2}{2} = 1$
 $1^2 = 1$

Complete the square for $x^2 + 2x$

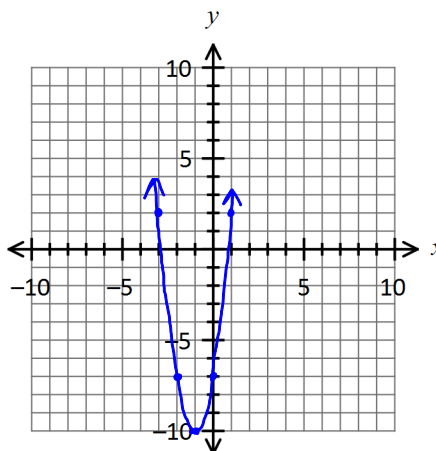
$f(x) = 3(x^2 + 2x + 1) - 7 - 3$

Actually added $3(1) = 3$

subtract 3 to have net effect of adding zero.

$f(x) = 3(x + 1)^2 - 10$

vertex: $(-1, -10)$
 axis of symmetry: $x = -1$
 minimum value = -10



d) $y = (-2x^2 + 12x) - 13$

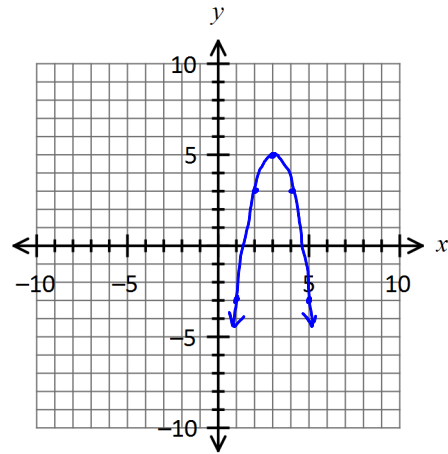
$y = -2(x^2 - 6x) - 13$

$-\frac{b}{2a} = -3$
 $(-3)^2 = 9$

$y = -2(x^2 - 6x + 9) - 13 + 18$

$y = -2(x - 3)^2 + 5$

Vertex: (3, 5)
 axis of symmetry: $x = 3$
 maximum value = 5



e) $f(x) = \frac{1}{2}x^2 + 6x + 15$

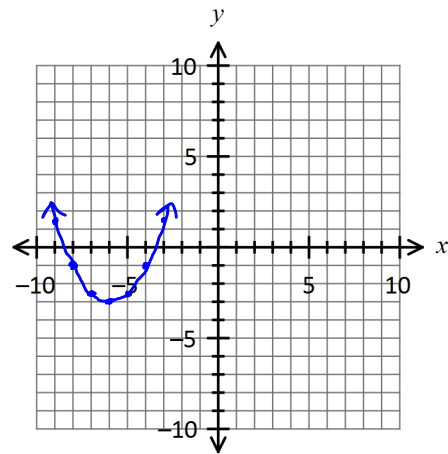
$f(x) = \frac{1}{2}(x^2 + 12x) + 15$

$\frac{b}{2a} = 6$
 $6^2 = 36$

$f(x) = \frac{1}{2}(x^2 + 12x + 36) + 15 - 18$

$f(x) = \frac{1}{2}(x + 6)^2 - 3$

Vertex: (-6, -3)
 axis of symmetry: $x = -6$
 minimum value = -3



The Vertex Formula

By completing the square, we can rewrite $f(x) = ax^2 + bx + c$ as $f(x) = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$.

This gives us a quick way to find the vertex when the equation is in standard form:

- The x -coordinate of the vertex is $-\frac{b}{2a}$.
- To find the y -coordinate, plug the x -coordinate into the original equation.

Properties of the Graph of a Quadratic Function

For the function, $f(x) = ax^2 + bx + c$, $a \neq 0$:

Vertex: $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$

Axis of Symmetry: The line $x = \frac{-b}{2a}$

Parabola opens up if $a > 0$; the vertex is a minimum point.

Parabola opens down if $a < 0$; the vertex is a maximum point.

Examples: Use the vertex formula to locate the vertex and axis of symmetry of the parabola. Determine whether the quadratic function has a maximum or minimum value, then find the value.

a) $f(x) = x^2 - 6x - 8$

b) $f(x) = -4x^2 + 2x + 1$

$$x = \frac{-b}{2a} = \frac{6}{2(1)} = \frac{6}{2} = 3$$

$$f(3) = 3^2 - 6(3) - 8 = 9 - 18 - 8 = -17$$

Vertex: (3, -17)
axis of symmetry: $x = 3$
minimum value = -17

$$x = \frac{-b}{2a} = \frac{-2}{2(-4)} = \frac{-2}{-8} = \frac{1}{4}$$

$$f\left(\frac{1}{4}\right) = -4\left(\frac{1}{4}\right)^2 + 2\left(\frac{1}{4}\right) + 1 = -4\left(\frac{1}{16}\right) + 2\left(\frac{1}{4}\right) + 1 = -\frac{1}{4} + \frac{1}{2} + 1 = \frac{5}{4}$$

Vertex: $\left(\frac{1}{4}, \frac{5}{4}\right)$
Axis of Symmetry: $x = \frac{1}{4}$
Maximum Value = $\frac{5}{4}$

Intercepts of Quadratic Graphs

The y-intercept is the value of f at $x = 0$; that is, the y-intercept is $f(0) = c$.

The x-intercepts, if any, are found by solving the quadratic equation $ax^2 + bx + c = 0$.

| Discriminant | Solutions of $ax^2 + bx + c = 0$ | Graph of $f(x) = ax^2 + bx + c$ |
|-----------------|----------------------------------|---------------------------------|
| $b^2 - 4ac > 0$ | Two real solutions | Two x-intercepts |
| $b^2 - 4ac = 0$ | One real solution | One x-intercept |
| $b^2 - 4ac < 0$ | Two imaginary solutions | No x-intercepts |

Examples: Find the x- and y-intercepts of the graph of each function.

a) $f(x) = 2x^2 + 7x - 30$

b) $y = x^2 - 6x + 4$

c) $f(x) = -3x^2 + 8x - 7$

y-int: $f(0) = 2(0)^2 + 7(0) - 30 = -30$

y-int: $y = 0^2 - 6(0) + 4 = 4$

y-int: $f(0) = -3(0)^2 + 8(0) - 7 = -7$

x-ints: $0 = 2x^2 + 7x - 30$

x-ints: $0 = x^2 - 6x + 4$

x-ints: $x = \frac{-8 \pm \sqrt{8^2 - 4(-3)(-7)}}{2(-3)}$

$$0 = 2x^2 + 12x - 5x - 30$$

$$0 = 2x(x+6) - 5(x+6)$$

$$x = \frac{6 \pm \sqrt{(-6)^2 - 4(1)(4)}}{2(1)}$$

$$x = \frac{-8 \pm \sqrt{-20}}{-6} \leftarrow \text{imaginary}$$

$$0 = (x+6)(2x-5)$$

$$x = -6 \text{ or } x = \frac{5}{2}$$

$$x = \frac{6 \pm \sqrt{20}}{2} = \frac{6 \pm 2\sqrt{5}}{2}$$

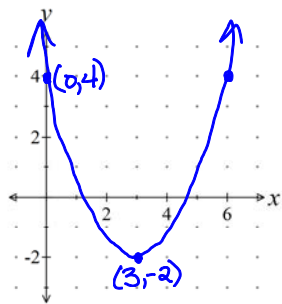
$$x = 3 + \sqrt{5}, x = 3 - \sqrt{5}$$

no x-intercepts

Writing a Quadratic Equation when You Know the Vertex and Another Point

- Use vertex form: $y = a(x-h)^2 + k$
- Plug in the vertex for h and k .
- Plug in the other point for x and y (or $f(x)$).
- Simplify and solve for a . (Don't forget order of operations.)
- Write your final answer by plugging a , h , and k back into vertex form.

Example: Find the quadratic function whose vertex is $(3, -2)$ and whose y-intercept is 4. Graph the function.



$$y = a(x-h)^2 + k$$

vertex: $(3, -2)$

$$y = a(x-3)^2 - 2$$

goes through $(0, 4)$

$$4 = a(0-3)^2 - 2$$

$$4 = a(-3)^2 - 2$$

$$4 = 9a - 2$$

$$6 = 9a$$

$$a = \frac{6}{9} = \frac{2}{3}$$

$$y = \frac{2}{3}(x-3)^2 - 2$$

| | Standard Form | Vertex Form | Factored Form |
|------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Equation | $y = ax^2 + bx + c$ | $y = a(x-h)^2 + k$ | $y = a(x-p)(x-q)$ |
| Vertex | Complete the square and write in vertex form. -or- $x = \frac{-b}{2a}$ Plug the x -coordinate into the equation to get the y -coordinate. | (h, k) (opposite of # with x , # at end) | Find average of p and q . $x = \frac{p+q}{2}$ (The x -coordinate of the vertex is at the midpoint of the x -intercepts.) Plug the x -coordinate into the equation to get the y -coordinate. |
| y-intercept | c (Replace x with zero. Solve for y .) | Replace x with zero. Solve for y . | Replace x with zero. Solve for y . |
| x-intercepts (roots, zeros) | Replace y with zero. Solve for x by factoring or quadratic formula. | Replace y with zero. Solve for x by isolating the perfect square and using the square root principle. (Don't forget the \pm .) | p and q (Replace y with zero. Solve for x using the zero product property.) |

For all forms:

| | |
|------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Direction of Opening | Up if a is positive Down if a is negative |
| Vertical Stretch | a |
| Counting Pattern (Shortcut) | Start at the vertex. Find more points by counting: $\leftrightarrow 1, \uparrow a$ $\leftrightarrow 1, \uparrow 3a$ $\leftrightarrow 1, \uparrow 5a$ $\leftrightarrow 1, \uparrow 7a$, etc. (If a is negative, move down instead of up.) |