

## Polar coordinates and Polar Graphs

When we first learned about vectors we learned that vectors had magnitude and direction. We then had position vectors that told us how far the particle was from the origin and in which direction. As it turns out there is a coordinate system that is set up for this type of tracking system. It is called the Polar Coordinate System. What does it look like?

### Polar Coordinates

$P(r, \theta)$

$r$  is the directed distance from  $O$  to  $P$      $\theta$  is the directed angle from initial ray to ray  $OP$

### Example 1

### Polar Graphing

Equation      Polar Graph

$r = a$       Circle of radius  $|a|$  centered at  $O$ .

$\theta = a$       Line through  $O$  making an angle  $a$  with the initial ray.

**Examples:**     $r = 1$                                    $r = 1.5$

**Examples:**                       $\theta = \frac{\pi}{4}$                        $\theta = \frac{\pi}{3}$                        $\theta = -\frac{3\pi}{4}$

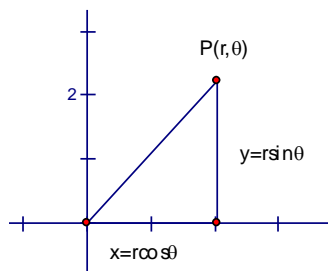
### Example 2 Graphing with Polar Coordinates

There are much more interesting polar functions

**Example:**  $r = 1 - \sin \theta$     (Graph by hand)

**Example 3:**     $r = \sin 6\theta$                        $r = 1 - 2\cos\theta$                        $r = 4\sin\theta$

Without much difficulty we can show how polar graphs relate to Cartesian graphs. Polar graphs have a huge advantage in that they can graph things that don't appear to be functions in the Cartesian form. To make the conversion, we need only picture any point on a polar graph.



There are other relationships that naturally arise. Using the theorem of Pythagoras we get.

$$x = r \cos \theta \quad \text{and} \quad y = r \sin \theta \quad \text{then} \quad x^2 + y^2 = r^2$$

$$\text{Also using trig we get } \tan \theta = \frac{y}{x} \quad \text{or} \quad \theta = \tan^{-1} \left( \frac{y}{x} \right)$$

#### Example 4:

$$r = 4 \sin \theta$$

$$r \cos \theta = 2$$

$$r^2 \cos \theta \sin \theta = 4$$

$$r^2 \cos^2 \theta - r^2 \sin^2 \theta = 1$$

#### The Parametric Equations of Polar Curves

The polar graph of  $r = f(\theta)$  is the curve defined parametrically by:

$$x = r \cos(\theta) = f(\theta) \cos(\theta) \quad \text{and} \quad y = r \sin(\theta) = f(\theta) \sin(\theta)$$

Since polar curves are drawn in the  $xy$ -plane, the slope of a polar curve is still the slope of the tangent line, which is  $\frac{dy}{dx}$ . The polar-rectangular conversion formulas enable us to write  $x$  and  $y$

as functions of  $\theta$ , so we can find  $\frac{dy}{dx} \cdot \frac{dx}{d\theta} = \frac{dy}{d\theta}$ . If  $f$  is a differentiable function of  $\theta$ , then so are  $x$  and  $y$  and, when  $dx/d\theta \neq 0$ , we can calculate  $dy/dx$  from the parametric formula.

#### Slope of the curve $r = f(\theta)$

$$\left. \frac{dy}{dx} \right|_{(r,\theta)} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{f'(\theta) \sin \theta + f(\theta) \cos \theta}{f'(\theta) \cos \theta - f(\theta) \sin \theta}$$

We should note that special circumstances **do** arise.

1. Horizontal tangent at a point where  $dy/d\theta = 0$  and  $dx/d\theta \neq 0$
2. Vertical tangent at a point where  $dx/d\theta = 0$  and  $dy/d\theta \neq 0$
3. Inconclusive when  $dx/d\theta = dy/d\theta = 0$

### Example 5

39. Find the slope of the curve at each indicated point.  $r = -1 + \sin\theta, \theta = 0, \pi$

Area in Polar Coordinates: The area of the region between the origin and the curve  $r = f(\theta)$  for  $\alpha \leq \theta \leq \beta$  is  $A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta = \int_{\alpha}^{\beta} \frac{1}{2} (f(\theta))^2 d\theta$

### Example 6

### Example 7

Area Between Polar Curves: The area of the region between  $r_1(\theta)$  and  $r_2(\theta)$  for  $\alpha \leq \theta \leq \beta$  is  $A = \int_{\alpha}^{\beta} \frac{1}{2} r_2^2 d\theta - \int_{\alpha}^{\beta} \frac{1}{2} r_1^2 d\theta = \int_{\alpha}^{\beta} \frac{1}{2} (r_2^2 - r_1^2) d\theta$

### Example 8

53. Find the area of the region shared by the circle  $r = 2$  and the cardioid  $r = 2(1 - \cos\theta)$

73. Length of a Polar Curve  $L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$