## Binomial and Geometric Probability

When the same chance process is repeated several times, we are often interested in whether a particular outcome does or doesn't happen on each repetition.

A ***binomial setting*** is one in which the following conditions are met:

- **Binary:** All possible outcomes of each trial can be classified as either "successes" or "failures".
- **Independent:** The trials are independent. Knowing what happens on one trial must not tell us anything about how likely a success is on any other trial.
- **Fixed probability of success:** The probability of success, $p$, is the same for each trial.
- **Counting the number of successes ($k$) in a fixed number of trials ($n$).**

**Examples:** Determine whether each setting is a binomial setting. Explain.

a) Roll a fair die 10 times and count how many times you roll a 6.

Binary? success: 6 failure: anything else

Independent? Knowing outcome of one roll gives no insight into other rolls

Fixed prob. of success: $p = 1/6$

Fixed # of trials: $n = 10$

binomial

b) Shoot a basketball 20 times from various distances on the court. Count how many shots you make.

Independent: Debatable – hot hand effect?

Fixed prob.: No! Shooting from various distances

not binomial

c) Shuffle a deck of cards. Turn over the top five cards one at a time and count the number of hearts.

Independent – No! Knowing one card changes prob. next one is hearts

not binomial

d) Flip a coin until you flip heads. Count how many flips it takes.

Don't have fixed # of trials

not binomial

**Example:** The probability of rolling doubles when rolling two dice is $6/36$ or $1/6$. If you are playing a game in which rolling doubles is beneficial, what is the probability that you will roll doubles on exactly 2 of your next 5 turns?

a) List all the possible ways you could end up with doubles on exactly 2 of your next 5 turns. How many different ways are there for that to happen?

.0161 → D D N N N    N D D N N    N N D N D
.0161 → D N D N N    N D N D N    N N N D D
.0161 → D N N D N    N D N N D
.0161 → D N N N D    N N D D N

10 ways

b) Is there an easier way to figure out how many ways you could end up with doubles on 2 of your next 5 turns? How? Verify that you have the correct number of possibilities.

Combination!

$${}_5C_2 = \frac{5!}{3!\,2!} = 10$$

c) The probability of each of the specific possibilities you listed in a) is the same. Calculate this probability.

D D N N N

$$\left(\tfrac{1}{6}\right)\left(\tfrac{1}{6}\right)\left(\tfrac{5}{6}\right)\left(\tfrac{5}{6}\right)\left(\tfrac{5}{6}\right) = \left(\tfrac{1}{6}\right)^2\left(\tfrac{5}{6}\right)^3 \approx .0161$$

d) Combine your answers from the steps above to calculate the probability of rolling doubles on exactly 2 of your next 5 turns.

$$10\left(\tfrac{1}{6}\right)^2\left(\tfrac{5}{6}\right)^3 \approx \boxed{.161}$$

***Binomial Probability:*** If $X$ has the binomial distribution with $n$ trials and probability $p$ of success on each trial, then the possible values of $X$ are $0, 1, 2, \ldots, n$. If $k$ is any one of these values, then

$$P(X=k) = ({}_nC_k)\,p^k(1-p)^{n-k} \quad \text{or} \quad P(X=k) = ({}_nC_k)\,p^k q^{n-k}\text{, where}$$

$n$ = number of trials    $p$ = probability of success on each trial

$k$ = number of successes    $q = 1-p$ = probability of failure on each trial

★ **Note:** I find it easier to think of this formula in words:

$$\left({}_{\text{\# of trials}}C_{\text{\# of successes}}\right)(\text{probability of success})^{\text{\# of successes}}(\text{probability of failure})^{\text{\# of failures}}$$

**Binomial Probabilities on a Calculator**

The "Distribution" menu ($2^{nd}$ Vars) on a TI-83 or TI-84 has two commands involving binomial probabilities:

**binompdf** $(n, p, k)$ gives $P(X=k)$, the probability of **exactly** $k$ **successes** in $n$ trials.

**binomcdf** $(n, p, k)$ gives $P(X \le k)$, the probability of $k$ **or fewer successes** in $n$ trials.

**Example:** Mr. Meany-Pants gives his class a 10-question multiple choice quiz. Each question has 5 possible answers. However, all of the questions and answer choices are written in Arabic, and none of the students speak Arabic, so they are forced to guess on every question. Mr. Meany-Pants used a random number generator to decide where to put the correct answer, and each answer has an equal chance of being correct. George is one of the students in the class. Let $X$ = the number of questions George answers correctly.

a) Find the probability that George will get exactly 3 questions right.

$P(X=3) = ({}_{10}C_3)\left(\frac{1}{5}\right)^3\left(\frac{4}{5}\right)^7 \approx .2013$

or binomialpdf $(n=10,\ p=\frac{1}{5},\ k=3) \approx .2013$

b) Find the probability that George gets at most 3 questions right.

$\leq 3$ 0 1 2 or 3

$P(X\leq 3) = ({}_{10}C_0)\left(\frac{1}{5}\right)^0\left(\frac{4}{5}\right)^{10} + ({}_{10}C_1)\left(\frac{1}{5}\right)^1\left(\frac{4}{5}\right)^9 + ({}_{10}C_2)\left(\frac{1}{5}\right)^2\left(\frac{4}{5}\right)^8 + ({}_{10}C_3)\left(\frac{1}{5}\right)^3\left(\frac{4}{5}\right)^7$

$\approx 0.8791$ or

binomialcdf $(n=10,\ p=\frac{1}{5},\ k=3) \approx 0.8791$

c) To get a passing score on the quiz, a student must get more than half of the problems right. What is the probability that George will pass the quiz? $> 5$

k

0 1 2 3 4 5 6 7 8 9 10

$P(X>5) = 1 - P(X\leq 5)$ = 1 − binomialcdf $(n=10,\ p=\frac{1}{5},\ k=5)$

$\approx \boxed{.0064}$

**Example:** A pet food manufacturer is promoting a new brand with a rebate offer on its 10-pound bag. Each package is supposed to contain a coupon for a \$4.00 mail-in rebate. The company has found that the machine dispensing these coupons fails to place a coupon in 15% of the bags. A dog owner buys 5 bags. Let $X$ = the number of bags bought by the dog owner that don't contain a coupon.

Success = no coupon

$n=5$ $p=.15$ prob of "success"

a) Find $P(X=1)$. What does this mean in context?

P(exactly one bag doesn't have coupon)

$P(X=1) = ({}_5C_1)(.15)^1(.85)^4 \approx .3915$

or

binomialpdf $(n=5,\ p=.15,\ k=1)$

b) Find the probability that less than three bags will be missing a coupon.

$P(X<3) = P(X\leq 2)$ = binomialcdf $(n=5,\ p=.15,\ k=2) \approx .9734$

0 1 2 3 4 5

c) Find the probability that at least two bags will be missing a coupon.

$P(X\geq 2) = 1 - P(X\leq 1)$ = 1 − binomialcdf $(n=5,\ p=.15,\ k=1) = .16479$

0 1 2 3 4 5

k

## Geometric Random Variables

A ***geometric setting*** is one in which we perform independent trials of the same chance process and record the number of trials until a particular outcome occurs. The following are the conditions are met:

- **Binary:** All possible outcomes of each trial can be classified as either "successes" or "failures".
- **Independent:** The trials are independent. Knowing the result of one trial must not tell us anything about the result of any other trial.
- **Fixed probability of success:** The probability of success, $p$, is the same for each trial.
- **Counting the number of trials to the first success:** The goal is to count the number of trials until the first success.

★ **Notice that the first three conditions are the same as those for a binomial setting. The only difference is that the goal is different. In a binomial setting, we count the number of successes in a fixed number of trials. In a geometric setting, we count the number of trials before the first success.**

***Geometric Probability:*** The number of trials $Y$ that it takes to get a success in a geometric setting is a ***geometric random variable.*** The probability distribution of $Y$ is a ***geometric distribution.*** A geometric distribution is completely defined by giving the value of $p$, the probability of success on each trial. The possible values of $Y$ are 1, 2, 3, … If $k$ is any one of these values,

$$P(Y=k)=(1-p)^{k-1}p \quad \text{or} \quad P(Y=k)=q^{k-1}p$$

## Geometric Probabilities on a Calculator

The "Distribution" menu (2nd Vars) on a TI-83 or TI-84 has two commands involving geometric probabilities:

**geometpdf** (***p***, ***k***) gives $P(X=k)$, the probability that **the first success happens *on* the $k^{th}$ trial.**

**geometcdf** (***p, k***) gives $P(X \le k)$, the probability that **the first success happens *on or before* the $k^{th}$ trial.**

**Example:** About 44% of people in the U.S. have Type O blood. Suppose we select people at random until we find someone with Type O blood.

a) Find the probability that the third person selected will be the first person with Type O blood.

x = 1st w/ type O

F F S

$P(X=3) = (.56)(.56)(.44) \approx .1380$

or geometpdf $(p=.44, k=3) \approx .1380$

| First person w/ Type O | prob |
|---|---|
| 1 | 0.44 |
| 2 | $(.56)(.44)$ |
| 3 | $(.56)^2(.44)$ |
| 4 | $(.56)^3(.44)$ |
| 5 | $(.56)^4(.44)$ |

geometric sequence

b) Find the probability that we will have to select less than 5 people to find someone with Type O blood.

$P(X<5) = P(X \le 4) =$ geometcdf $(p=.44, k=4) \approx .9017$

1 2 3 4 5 . . .

or $1 - P(\text{none of 1st 4 has Type O}) =$ 1 − F F F F

$1 - (.56)^4 \approx .9017$

c) Find the probability that we will have to select more than 3 people to find the first person with Type O blood.

F F F

$P(X>3) = P(\text{none of 1st 3 had Type O}) = (.56)^3 \approx .1756$

$1 - P(X \le 3) = 1 -$ geometcdf $(p=.44, k=3) \approx .1756$

1 2 3 4 5 . . .