Probability with Counting Rules

The ***sample space S*** of a chance process is the set of all possible outcomes.

Example: Give the sample space for flipping a coin three times.

{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

Example: Give the sample space for rolling two dice.

Die 1

Die 2						
	1,1	2,1	3,1	4,1	5,1	6,1
	1,2	2,2	3,2	4,2	5,2	6,2
	1,3	2,3	3,3	4,3	5,3	6,3
	1,4	2,4	3,4	4,4	5,4	6,4
	1,5	2,5	3,5	4,5	5,5	6,5
	1,6	2,6	3,6	4,6	5,6	6,6

Event: Any outcome or collection of outcomes from some chance process is called an ***event.*** An event is a subset of the sample space. Events are usually designated by capital letters. We write the probability of event A as $P(A)$.

Chance Process	Sample Space	Event
Flip a coin	$S = \{\text{heads, tails}\}$	$B = \{\text{heads}\}$
Roll a die	$S = \{1,\ 2,\ 3,\ 4,\ 5,\ 6\}$	Even numbers $E = \{2,\ 4,\ 6\}$
Pick a letter in the word "probability"	$S = \{\text{P, R, O, B, A, I, L, T, Y}\}$	Vowels $V = \{\text{O, A, I, Y}\}$

If all outcomes in the sample space are equally likely, the ***probability*** that event A occurs is

$$P(A) = \frac{\text{\# of outcomes corresponding to event } A}{\text{total \# of outcomes in the sample space}}$$

★ **Note:** This formula only works if all the outcomes in the sample space are equally likely, which is not always true!

Basic Rules of Probability

- *The probability of any event is a number between 0 and 1.*
  - *For any event A,* $0 \le P(A) \le 1$.
  - The probability of an event is the long-run proportion of repetitions on which that event occurs. An event with probability 0 never occurs, and an event with probability 1 occurs on every trial.
- *All possible outcomes of a chance process must have probabilities whose sum is 1.*
- *The probability that an event does not occur is 1 minus the probability that the event does occur.*
  - $P(A^C) = 1 - P(A)$.
  - Since it is certain that an event will either occur or not occur, the probability that an event occurs and the probability that it doesn't occur always add to 100% or 1.

Many probability problems come down to counting the number of ways an event can happen and the number of outcomes in the sample space. This can either be done by making a list of the sample space or by using the Fundamental Counting Rule, permutations, and combinations.

Example: If you flip a coin 3 times, what's the probability of flipping heads at least twice? (Solve by looking at the sample space you listed on the previous page. We'll learn how to calculate this probability with formulas later.)

{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

$$\frac{4}{8} = \boxed{\frac{1}{2}}$$

Example: If you roll a pair of dice, what's the probability the sum will be 8 or 9?

Die 1

Die 2	1	2	3	4	5	6
1	1,1	2,1	3,1	4,1	5,1	6,1
2	1,2	2,2	3,2	4,2	5,2	6,2
3	1,3	2,3	3,3	4,3	5,3	6,3
4	1,4	2,4	3,4	4,4	5,4	6,4
5	1,5	2,5	3,5	4,5	5,5	6,5
6	1,6	2,6	3,6	4,6	5,6	6,6

$$\frac{9}{36} = \boxed{\frac{1}{4}}$$

Example: If license plate numbers consisting of three letters followed by three numbers are assigned at random, what is the probability that the plate number will end with 911? (Assume letters and numbers can be repeated and all sequences are allowed.)

$\underbrace{L\ L\ L}_{26\times26\times26 \text{ possibilities for this part}}\ \underbrace{9\ 1\ 1}_{\text{only one choice for this part}}$ ← 17,576

L L L # # #

$26\times26\times26\times10\times10\times10 = 17{,}576{,}000$

$$P(\text{ends in } 911) = \frac{17{,}576}{17{,}576{,}000} = \boxed{\frac{1}{1000} = 0.001}$$

Example: A vocabulary section on a test consists of 6 words that need to be matched with their definitions. If you have no idea what any of the words mean and are completely guessing, what is the probability that you will match all 6 of the words with the correct definitions?

$6!$ or ${}_6P_6 = 720$ arrangements of answers

Only 1 is correct

$$P(\text{all 6 match}) = \boxed{\frac{1}{720} \approx 0.00139}$$

Examples: A bag contains 1 green, 4 red, and 5 yellow balls. ← 10 total. Two balls ← choosing 2 are selected at random. Find the probability of each selection. Express your answers as simplified fractions.

${}_{10}C_2 = 45$ possible choices of 2 balls from 10

a. $P(2 \text{ red})$

$\dfrac{{}_4C_2}{{}_{10}C_2}$ ← choosing 2 of the 4 red balls

$= \dfrac{6}{45} = \boxed{\dfrac{2}{15}}$

b. $P(1 \text{ red and } 1 \text{ yellow})$

$\dfrac{{}_4C_1 \cdot {}_5C_1}{{}_{10}C_2}$ ← choosing 1 of 4 red & 1 of 5 yellow

$= \dfrac{20}{45} = \boxed{\dfrac{4}{9}}$

c. $P(1 \text{ green and } 1 \text{ yellow})$

$\dfrac{{}_1C_1 \cdot {}_5C_1}{{}_{10}C_2}$ ← choosing 1 of 1 green & 1 of 5 yellow

$= \dfrac{5}{45} = \boxed{\dfrac{1}{9}}$

d. $P(2 \text{ yellow})$

$\dfrac{{}_5C_2}{{}_{10}C_2}$ ← choosing 2 of 5 yellow

$= \dfrac{10}{45} = \boxed{\dfrac{2}{9}}$

e. $P(1 \text{ red and } 1 \text{ green})$

$\dfrac{{}_4C_1 \cdot {}_1C_1}{{}_{10}C_2}$ ← choosing 1 of 4 red & 1 of 1 green

$= \boxed{\dfrac{4}{45}}$

Example: A drawer contains 3 blue pens and 7 black pens. I reach in and draw 5 pens at random. What is the probability that exactly 2 of the pens are blue? The other 3 must be black

$({}_3C_2)({}_7C_3)$ ← number of different ways to choose 2 of 3 blue pens & 3 of 7 black pens

$= 105$

${}_{10}C_5 = 252$

↑ total # of possible groups of 5 pens that can be chosen from 10

$P(2 \text{ blue}, 3 \text{ black}) = \dfrac{105}{252} = \boxed{\dfrac{5}{12}}$

$\approx \boxed{0.417}$

Example: A small voting district has 101 female voters and 95 male voters. A random sample of 10 voters is drawn. What is the probability exactly 7 of the voters will be female? (3 male)

196 total

$$P(7F, 3M) = \frac{({}_{101}C_7)({}_{95}C_3)}{{}_{196}C_{10}} = \frac{2.381 \times 10^{15}}{1.826 \times 10^{16}} \approx \boxed{0.130}$$

Examples: A standard deck of playing cards contains 52 cards: 13 each of spades (black), clubs (black), hearts (red), and diamonds (red). The 13 cards in each suit are the 2, 3, 4, 5, 6, 7, 8, 9, 10, the three face cards: the jack (J), queen (Q), and king (K), and the ace (A).

a. If a hand of 5 cards is selected at random, what is the probability of a flush (all the same suit)?

choices of suit

choices of 5 of the 13 cards of one suit

$$P(\text{flush}) = \frac{4({}_{13}C_5)}{{}_{52}C_5} = \frac{5148}{2{,}598{,}960} \approx \boxed{0.00198}$$

← # of 5-card hands

b. What is the probability that a hand of 5 cards will include 4 aces?

choosing all 4 aces

choosing 1 of the other 48 cards

$$P(4 \text{ aces}) = \frac{({}_4C_4)({}_{48}C_1)}{{}_{52}C_5} = \frac{48}{2{,}598{,}960} \approx \boxed{0.0000185}$$

c. What is the probability that a hand of 7 cards will include 3 hearts, 2 diamonds, a club, and a spade?

$$P(3H, 2D, 1C, 1S) = \frac{({}_{13}C_3)({}_{13}C_2)({}_{13}C_1)({}_{13}C_1)}{({}_{52}C_7)} = \frac{3{,}770{,}052}{133{,}784{,}560} \approx \boxed{0.0282}$$

Example: Powerball is a multi-state lottery. Players choose five different numbers from 1 to 69 and one Powerball number from 1 to 26. Twice per week, 5 white balls are drawn randomly from a drum with 69 white balls, numbered 1 to 69, and then one red Powerball is drawn randomly from a drum with 26 red balls, numbered 1 to 26. A player wins the jackpot by matching all five numbers drawn from the white balls in any order and matching the number on the red Powerball. What is the probability of winning the jackpot with one $2 Powerball ticket?

$$P(\text{jackpot}) = \frac{1}{({}_{69}C_5)(26)} = \boxed{\frac{1}{292{,}201{,}338} \approx 0.00000000342}$$