The Fundamental Counting Principle, Permutations, and Combinations

Fundamental Counting Principle:

★ **If event *M* can occur in *m* ways and event *N* can occur in *n* ways, then event *M* followed by event *N* can occur in $m \cdot n$ ways.**

- If you are making a sandwich and there are three types of bread, five types of meat, and four types of cheese available and you choose one type of bread, one type of meat, and one type of cheese, there are $3 \cdot 5 \cdot 4 = 60$ different sandwiches you can make.

Examples:

a) How many possible locker combinations are available if each combination consists of three numbers between 1 and 50?

$\underline{50} \cdot \underline{50} \cdot \underline{50} = 125{,}000$

b) How many license plates consisting of three numbers followed by three letters are possible if numbers and letters can be repeated?

$\underline{10} \cdot \underline{10} \cdot \underline{10} \cdot \underline{26} \cdot \underline{26} \cdot \underline{26} = 17{,}576{,}000$

c) There are eight true-false questions on a quiz. How many different answer combinations are possible?

2 possibilities for each question

$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^8 = 256$

d) A golf club manufacturer makes drivers with 4 different shaft lengths, 3 different lofts, 2 different grips, and 2 different club head materials. How many different combinations are possible?

$\underline{4} \cdot \underline{3} \cdot \underline{2} \cdot \underline{2} = 48$

e) For a college application, George must select one of five topics on which to write a short essay. He must select a different topic from the list for a longer essay. How many ways can he choose the topics for the two essays?

$\underline{5} \cdot \underline{4} = 20$

choices of 1st topic

choices of 2nd topic

f) Abby is registering at a website. She must create a password containing 6 *different* numerical digits. How many possible passwords are there?

$\underline{10}\cdot\underline{9}\cdot\underline{8}\cdot\underline{7}\cdot\underline{6}\cdot\underline{5} = 151{,}200$

g) How many ways can 9 books be arranged on a shelf?

9 possible locations for first book, 8 for second book, 7 for third book, ..., 1 for ninth book

$9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1 = 9! = 362{,}880$

Permutation: An arrangement of a group of objects in a certain order. In a permutation, order matters.

Factorial: The symbol $n!$, which is read as "n factorial" is defined as follows:

$0!=1 \qquad 1!=1 \qquad 2!=2\cdot1 \qquad 3!=3\cdot2\cdot1 \qquad 4!=4\cdot3\cdot2\cdot1 \qquad n!=n\cdot(n-1)\cdot\ldots\cdot3\cdot2\cdot1$

★ **The number of ways of arranging *n* distinct objects is $n!$.**

Example: How many ways can you arrange the letters in BINGHAM?

7 letters

$7! = 5040$

★ **The number of ways of arranging *n* objects of which *p* of them are repeated and *q* of them are repeated is $\dfrac{n!}{p!q!}$. This rule can be extended to any number of repeated objects.**

Examples:

a) How many ways can you arrange the letters in ALGEBRA?

7 letters, 2 As

$\dfrac{7!}{2!} = 2520$

b) How many ways can you arrange the letters in MISSISSIPPI?

11 letters, 4 Is, 4 Ss, 2 Ps

$\dfrac{11!}{4!\,4!\,2!} = 34{,}650$

★ **The number of ways that a subgroup of *r* objects from a set of *n* distinct objects can be arranged is given by $_nP_r = \dfrac{n!}{(n-r)!}$.** This is also sometimes written as $P(n,r)$.

Examples: Calculate the following.

a) $_6P_2 = \dfrac{6!}{(6-2)!} = \dfrac{6!}{4!}$

$= \dfrac{6\cdot5\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}{\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}$

$= 30$

b) $_9P_4 = \dfrac{9!}{(9-4)!} = \dfrac{9!}{5!}$

$= \dfrac{9\cdot8\cdot7\cdot6\cdot\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}{\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}$

$= 3024$

c) $_8P_8 = \dfrac{8!}{(8-8)!} = \dfrac{8!}{0!} = \dfrac{8!}{1}$

$= 8! = 40{,}320$

★ **Notice that all permutation problems can also be solved using the Fundamental Counting Principal!**

Examples:

a) Sixteen teams are competing in a soccer tournament. Gold, silver, and bronze medals will be awarded to the top three teams. In how many different ways can the medals be awarded?

$${}_{16}P_3 = \frac{16!}{(16-3)!} = 3360 \quad \text{or} \quad \frac{16}{G} \cdot \frac{15}{S} \cdot \frac{14}{B} = 3360$$

b) A club with thirty members is selecting officers. In how many ways can they select a president, a vice president, a secretary, and a treasurer?

$${}_{30}P_4 = \frac{30!}{(30-4)!} = 657{,}720 \quad \text{or} \quad \frac{30}{P} \cdot \frac{29}{VP} \cdot \frac{28}{S} \cdot \frac{27}{T} = 657{,}720$$

c) In how many orders can a student read six books selected from a list of nine possibilities?

$${}_{9}P_6 = \frac{9!}{(9-6)!} = 60{,}480 \quad \text{or} \quad \underset{1st}{9} \cdot \underset{2nd}{8} \cdot \underset{3rd}{7} \cdot \underset{4th}{6} \cdot \underset{5th}{5} \cdot \underset{6th}{4} = 60{,}480$$

Combination: A selection of objects in which order is not important.

★ **The number of subgroups of r objects that can be selected from a group of n distinct objects is given by $_nC_r = \frac{n!}{(n-r)!r!}$.** This is sometimes written as $C(n,r)$ or $\binom{n}{r}$.

Examples: Calculate the following.

a) $_6C_2 = \frac{6!}{(6-2)!2!} = \frac{6!}{4!2!}$

$$= \frac{6 \cdot 5 \cdot \cancel{4 \cdot 3 \cdot 2 \cdot 1}}{\cancel{4 \cdot 3 \cdot 2 \cdot 1} \cdot 2 \cdot 1} = 15$$

b) $_9C_4 = \frac{9!}{(9-4)!4!} = \frac{9!}{5!4!}$

$$= \frac{9 \cdot \cancel{8} \cdot 7 \cdot 6 \cdot \cancel{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}}{\cancel{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \cdot \cancel{4} \cdot 3 \cdot \cancel{2} \cdot 1}$$

$$= 126$$

c) $_8C_8 = \frac{8!}{(8-8)!8!} = \frac{8!}{0!8!}$

$$= 1$$

Examples:

a) How many different groups of four students can be created from a class of 30 students?

$${}_{30}C_4 = \frac{30!}{26!4!} = 27{,}405$$

b) How many different hands of five cards can be drawn from a standard deck of 52 cards?

$${}_{52}C_5 = \frac{52!}{47!5!} = 2{,}598{,}960$$

c) In a standard deck of cards, there are four suits: clubs, diamonds, hearts, and spades. Each suit has 13 cards. How many hands of five cards consist of three clubs and two diamonds?

$$\underbrace{({}_{13}C_3)}_{\text{\# of groups of 3 clubs}} \cdot \underbrace{({}_{13}C_2)}_{\text{\# of groups of 2 diamonds}} = 22{,}308$$