Geometric Sequences and Geometric Series

A ***geometric sequence*** is one in which the ratio of successive terms is always the same nonzero number.

A ***geometric sequence*** may be defined recursively as $a_1 = a,\ a_n = ra_{n-1},$ where a is the first term and $r \neq 0$ is the ***common ratio***. The terms of a geometric sequence with first term a_1 and common ratio $r = \frac{a_n}{a_{n-1}}$ follow the pattern $a_1,\ a_1r,\ a_1r^2,\ a_1r^3, \ldots, a_1r^{n-1}$.

Examples: Find the common ratio of each geometric sequence and write out the first four terms.

a) $\{b_n\} = \{(-5)^n\}$

$b_1 = (-5)^1 = -5$ $\times(-5)$

$b_2 = (-5)^2 = 25$ $\times(-5)$

$b_3 = (-5)^3 = -125$ $\times(-5)$

$b_4 = (-5)^4 = 625$

$\boxed{r = -5}$

b) $\{u_n\} = \left\{\frac{2^n}{3^{n-1}}\right\}$

$u_1 = \frac{2^1}{3^{1-1}} = \frac{2^1}{3^0} = 2$ $\times\frac{2}{3}$

$u_2 = \frac{2^2}{3^{2-1}} = \frac{2^2}{3^1} = \frac{4}{3}$ $\times\frac{2}{3}$

$u_3 = \frac{2^3}{3^{3-1}} = \frac{2^3}{3^2} = \frac{8}{9}$ $\times\frac{2}{3}$

$u_4 = \frac{2^4}{3^{4-1}} = \frac{2^4}{3^3} = \frac{16}{27}$

$\boxed{r = \frac{2}{3}}$

★ To determine whether a sequence is arithmetic, geometric, or neither, find a_n and a_{n-1}. If $a_n - a_{n-1}$ is constant, the sequence is arithmetic. If $\frac{a_n}{a_{n-1}}$ is constant, the sequence is geometric.

Examples: Determine whether the given sequence is arithmetic, geometric, or neither. If it is arithmetic, give the common difference. If it is geometric, give the common ratio.

a) $\{5n^2 + 1\}$

$a_n = 5n^2 + 1$

$a_{n-1} = 5(n-1)^2 + 1$

$= 5(n^2 - 2n + 1) + 1$

$= 5n^2 - 10n + 5 + 1$

$= 5n^2 - 10n + 6$

arithmetic?

$a_n - a_{n-1} = (5n^2+1) - (5n^2 - 10n + 6)$

$= \cancel{5n^2} + 1 - \cancel{5n^2} + 10n - 6$

$= 10n - 5$

not constant
difference depends on which terms you are subtracting
not arithmetic

geometric?

$\frac{a_n}{a_{n-1}} = \frac{5n^2+1}{5n^2-10n+6}$ ← not constant
not geometric

$\boxed{\text{neither}}$

b) $\left\{\left(\frac{5}{4}\right)^n\right\}$

$a_n = \left(\frac{5}{4}\right)^n$

$a_{n-1} = \left(\frac{5}{4}\right)^{n-1}$

arithmetic?

$a_n - a_{n-1} = \left(\frac{5}{4}\right)^n - \left(\frac{5}{4}\right)^{n-1}$

not a constant
not arithmetic

geometric?

$\frac{a_n}{a_{n-1}} = \frac{\left(\frac{5}{4}\right)^n}{\left(\frac{5}{4}\right)^{n-1}} = \left(\frac{5}{4}\right)^{n-(n-1)} = \frac{5}{4}$

constant

$\boxed{\text{geometric, } r = \frac{5}{4}}$

***n*th Term of a Geometric Sequence:** For a geometric sequence $\{a_n\}$ whose first term is a_1 and whose common ratio is r, the nth term is determined by the formula $a_n = a_1r^{n-1}$, $r \neq 0$.

Examples: Find the nth term and the 5th term of the geometric sequence.

a) $a_1 = -2,\ r = 4$

$a_n = -2(4)^{n-1}$

$a_5 = -2(4)^{5-1} = \boxed{-512}$

b) $a_1 = 1,\ r = -\frac{1}{3}$

$a_n = \left(-\frac{1}{3}\right)^{n-1}$

$a_5 = \left(-\frac{1}{3}\right)^{5-1} = \boxed{\frac{1}{81}}$

Divide any term by the term before it to get r

Examples: Find the indicated term of each geometric sequence.

a) 8th term of 1, 3, 9, … $r=3$ ($\times 3$, $\times 3$)

$a_8 = 1(3)^{8-1} = \boxed{2187}$

b) 7th term of 9, –6, 4, … $r = -\frac{6}{9} = -\frac{2}{3}$

$a_7 = 9\left(-\frac{2}{3}\right)^{7-1} = \boxed{\frac{64}{81}}$

Examples: Find the *n*th term of each geometric sequence.

a) 5, 10, 20, 40, …

$a_1 = 5$ $r = 2$

$\boxed{a_n = 5(2)^{n-1}}$

b) $a_2 = 7,\ r = 1/4$ __, 7, $\frac{7}{4}$, …

$7 \div \left(\frac{1}{4}\right) = 7 \cdot 4 = 28$

OR $7 = a_1\left(\frac{1}{4}\right)^{2-1}$

$a_1 = \frac{7}{\left(\frac{1}{4}\right)^1} = 28$

$\boxed{a_n = 28\left(\frac{1}{4}\right)^{n-1}}$

c) $a_3 = 1/3,\ a_6 = 1/81$

$\times\frac{1}{27} \leftarrow \frac{1}{81} \div \frac{1}{3} = \frac{1}{81} \cdot \frac{3}{1}$

$\frac{3}{\ }, \frac{1}{\ }, \frac{1}{3}, \_\_, \_\_, \frac{1}{81}$ ($\times r$, $\times r$, $\times r$, $\times r$, $\times r$)

$r^3 = \frac{1}{27} \Rightarrow r = \frac{1}{3}$

$\frac{1}{3} = a_1\left(\frac{1}{3}\right)^{3-1} \Rightarrow a_1 = \frac{1/3}{(1/3)^2} = 3$ (or just reason it out)

$\boxed{a_n = 3\left(\frac{1}{3}\right)^{n-1}}$

Sum of the First *n* Terms of a Geometric Sequence

The sum S_n of the first *n* terms of a geometric sequence $\{a_n\}$ with first term a_1 and common ratio r is given by $S_n = \sum_{k=1}^{n} a_1 r^{k-1} = a_1 \cdot \frac{1-r^n}{1-r}$, where $r \neq 0,1$.

Examples: Find each sum.

a) $\frac{3}{9} + \frac{3^2}{9} + \frac{3^3}{9} + \ldots + \frac{3^n}{9}$

$a_1 = \frac{3}{9} = \frac{1}{3}$ $r = 3$ $n = n$

$S_n = \frac{1}{3}\left(\frac{1-3^n}{1-3}\right) = \frac{1}{3}\left(\frac{1-3^n}{-2}\right)$

$= \boxed{\frac{1-3^n}{-6}}$

b) $\sum_{k=1}^{15} 4 \cdot 3^{k-1}$ $\leftarrow n=15$

$r = 3$

$a_1 = 4 \cdot 3^{1-1} = 4$

$= 4\left(\frac{1-3^{15}}{1-3}\right) = 4\left(\frac{1-3^{15}}{-2}\right) = -2(1-3^{15})$

$= \boxed{28{,}697{,}812}$

c) $2 + \frac{6}{5} + \frac{18}{25} + \ldots + 2\left(\frac{3}{5}\right)^{15}$

$2\left(\frac{3}{5}\right)^0$ $2\left(\frac{3}{5}\right)^1$ $2\left(\frac{3}{5}\right)^2$

Adding 16 terms!

$n=16$

$r = \frac{3}{5}$

$S_{16} = 2\left(\frac{1-\left(\frac{3}{5}\right)^{16}}{1-\left(\frac{3}{5}\right)}\right) \approx \boxed{4.9986}$

d) $\sum_{k=1}^{5} \left(\frac{2}{3}\right)^k$ $\leftarrow n=5$

$r = \frac{2}{3}$

$a_1 = \left(\frac{2}{3}\right)^1 = \frac{2}{3}$

$\frac{2}{3}\left(\frac{1-\left(\frac{2}{3}\right)^5}{1-\frac{2}{3}}\right) = \boxed{\frac{422}{243}}$

Infinite Series*:** An infinite geometric series is the sum of the terms of an infinite geometric sequence. It is denoted by $\sum_{k=1}^{\infty} a_1 r^{k-1}$. If the sum of the first *n* terms of the geometric sequence, S_n, approaches a number L as $n \to \infty$, we say that the infinite geometric series ***converges and that its sum is L. If a series does not converge, it is a ***divergent series.***

$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots = 1$

If $|r| < 1$, the infinite geometric series $\sum_{k=1}^{\infty} a_1 r^{k-1}$ converges. Its sum is $S = \frac{a_1}{1-r}$.

r between –1 & 1

Examples: Determine whether each geometric series converges or diverges. If it converges, find its sum.

a) $2+\frac{4}{3}+\frac{8}{9}+\ldots$

$\times\frac{2}{3}$

$r=\frac{2}{3}$

converges

$S=\frac{2}{1-\frac{2}{3}}=\frac{2}{\frac{1}{3}}=\boxed{6}$

b) $9+12+16+\frac{64}{3}+\ldots$

$r=\frac{12}{9}=\frac{4}{3}$ $\boxed{\text{diverges}}$

c) $1-\frac{3}{4}+\frac{9}{16}-\frac{27}{64}+\ldots$

$r=-\frac{3}{4}$ converges

$S=\frac{1}{1-\left(-\frac{3}{4}\right)}=\frac{1}{\frac{7}{4}}=\boxed{\frac{4}{7}}$

d) $\sum_{k=1}^{\infty}3\left(\frac{3}{2}\right)^{k-1}$

$r=\frac{3}{2}$ $\boxed{\text{diverges}}$

e) $\sum_{k=1}^{\infty}4\left(-\frac{2}{5}\right)^{k}=\frac{-8/5}{1-\left(-\frac{2}{5}\right)}=\frac{-8/5}{7/5}=\boxed{-\frac{8}{7}}$

$r=-\frac{2}{5}$

converges

$a_1=4\left(-\frac{2}{5}\right)^1=-\frac{8}{5}$