Geometric Sequences and Geometric Series

A ***geometric sequence*** is one in which the ratio of successive terms is always the same nonzero number.

A ***geometric sequence*** may be defined recursively as $a_1 = a,\ a_n = ra_{n-1},$ where a is the first term and $r \neq 0$ is the ***common ratio*.** The terms of a geometric sequence with first term a_1 and common ratio $r = \dfrac{a_n}{a_{n-1}}$ follow the pattern $a_1,\ a_1r,\ a_1r^2,\ a_1r^3, \ldots, a_1r^{n-1}$.

Examples: Find the common ratio of each geometric sequence and write out the first four terms.

a) $\{b_n\} = \left\{(-5)^n\right\}$

b) $\{u_n\} = \left\{\dfrac{2^n}{3^{n-1}}\right\}$

★ To determine whether a sequence is arithmetic, geometric, or neither, find a_n and a_{n-1}. If $a_n - a_{n-1}$ is constant, the sequence is arithmetic. If $\dfrac{a_n}{a_{n-1}}$ is constant, the sequence is geometric.

Examples: Determine whether the given sequence is arithmetic, geometric, or neither. If it is arithmetic, give the common difference. If it is geometric, give the common ratio.

a) $\left\{5n^2 + 1\right\}$

b) $\left\{\left(\dfrac{5}{4}\right)^n\right\}$

***n*th Term of a Geometric Sequence:** For a geometric sequence $\{a_n\}$ whose first term is a_1 and whose common ratio is $r,$ the nth term is determined by the formula $a_n = a_1r^{n-1};\ r \neq 0.$

Examples: Find the *n*th term and the 5th term of the geometric sequence.

a) $a_1 = -2,\ r = 4$

b) $a_1 = 1,\ r = -\dfrac{1}{3}$

Examples: Find the indicated term of each geometric sequence.

a) 8th term of 1, 3, 9, …

b) 7th term of 9, –6, 4, …

Examples: Find the *n*th term of each geometric sequence.

a) 5, 10, 20, 40, …

b) $a_2 = 7,\ r = 1/4$

c) $a_3 = 1/3,\ a_6 = 1/81$

Sum of the First *n* Terms of a Geometric Sequence

The sum S_n of the first *n* terms of a geometric sequence $\{a_n\}$ with first term a_1 and common ratio r is given by $S_n = \sum_{k=1}^{n} a_1 r^{k-1} = a_1 \cdot \frac{1-r^n}{1-r}$, where $r \neq 0,1$.

Examples: Find each sum.

a) $\frac{3}{9} + \frac{3^2}{9} + \frac{3^3}{9} + \ldots + \frac{3^n}{9}$

b) $\sum_{k=1}^{15} 4 \cdot 3^{k-1}$

c) $2 + \frac{6}{5} + \frac{18}{25} + \ldots + 2\left(\frac{3}{5}\right)^{15}$

d) $\sum_{k=1}^{5} \left(\frac{2}{3}\right)^{k}$

Infinite Series*:** An infinite geometric series is the sum of the terms of an infinite geometric sequence. It is denoted by $\sum_{k=1}^{\infty} a_1 r^{k-1}$. If the sum of the first *n* terms of the geometric sequence, S_n, approaches a number L as $n \to \infty$, we say that the infinite geometric series ***converges and that its sum is L. If a series does not converge, it is a ***divergent series.***

If $|r| < 1$, the infinite geometric series $\sum_{k=1}^{\infty} a_1 r^{k-1}$ converges. Its sum is $S = \frac{a_1}{1-r}$.

Examples: Determine whether each geometric series converges or diverges. If it converges, find its sum.

a) $2+\frac{4}{3}+\frac{8}{9}+\ldots$

b) $9+12+16+\frac{64}{3}+\ldots$

c) $1-\frac{3}{4}+\frac{9}{16}-\frac{27}{64}+\ldots$

d) $\sum_{k=1}^{\infty} 3\left(\frac{3}{2}\right)^{k-1}$

e) $\sum_{k=1}^{\infty} 4\left(-\frac{2}{5}\right)^{k}$