Arithmetic Sequences

An ***arithmetic sequence*** is one in which the difference between successive terms of a sequence is always the same number.

An arithmetic sequence may be defined recursively as $a_1 = a,\ a_n = a_{n-1} + d$, where a is the first term and d is the ***common difference***. The terms of an arithmetic sequence with first term a_1 and common difference d follow the pattern $a_1,\ a_1 + d,\ a_1 + 2d,\ a_1 + 3d, \ldots,\ a_1 + (n-1)d$, where $d = a_n - a_{n-1}$.

Examples: Determine whether the following sequences are arithmetic:

a) $3, 7, 11, 15, 19, \ldots$

+4 +4 +4 +4

arithmetic

$d = 4$

b) $\frac{7}{3}, \frac{5}{3}, 1, \frac{1}{3}, \ldots$

$-\frac{2}{3}$ $-\frac{2}{3}$ $-\frac{2}{3}$

arithmetic

$d = -\frac{2}{3}$

★ To show that a sequence is arithmetic, find a_n and a_{n-1}. If $a_n - a_{n-1}$ is a constant (does not have a variable), then the sequence is arithmetic.

1. Plug in n
2. Plug in n−1
3. Subtract results

Examples: Show that the following sequences are arithmetic and find the common difference.

a) $\{s_n\} = \{2n - 4\}$

$S_n = 2n - 4$

$S_{n-1} = 2(n-1) - 4$
$= 2n - 2 - 4$
$= 2n - 6$

$S_n - S_{n-1} = (2n-4) - (2n-6) = 2n - 4 - 2n + 6 = 2$

constant (no n).
arithmetic
d=2

b) $\{b_n\} = \{\ln 2^n\}$

$b_n = \ln 2^{n} = n \ln 2$

$b_{n-1} = \ln 2^{n-1} = (n-1)\ln 2 = n\ln 2 - \ln 2$

$b_n - b_{n-1} = n\ln 2 - (n\ln 2 - \ln 2) = \ln 2$

constant.

arithmetic. $d = \ln 2$

***n*th Term of an Arithmetic Sequence:** For an arithmetic sequence $\{a_n\}$ whose first term is a_1 and whose common difference is d, the nth term is determined by the formula $a_n = a_1 + (n-1)d$.

Examples: Find the nth term and the fifty-first term of the following sequences.

a) $a_1 = 6,\ d = -2$

$a_n = 6 + (n-1)(-2)$
$a_n = 6 - 2n + 2$
$\boxed{a_n = -2n + 8}$

$a_{51} = 6 + (51-1)(-2)$
$= \boxed{-94}$

b) $a_1 = 1,\ d = -1/3$

$a_n = 1 + (n-1)(-\frac{1}{3})$
$a_n = 1 - \frac{1}{3}n + \frac{1}{3}$
$\boxed{a_n = -\frac{1}{3}n + \frac{4}{3}}$

$a_{51} = 1 + (51-1)(-\frac{1}{3})$
$= 1 - \frac{50}{3} = \boxed{-\frac{47}{3}}$

Examples: Find the indicated term in each arithmetic sequence.

a) 80th term of $29, 26, 23, 20, \ldots$

$n = 80$

−3 −3 −3

$d = -3$
$a_1 = 29$

$a_{80} = 29 + (80-1)(-3)$
$= \boxed{-208}$

b) 86th term of $2, \frac{5}{2}, 3, \frac{7}{2}, \ldots$

$d = \frac{1}{2}$

$n = 86$

$a_1 = 2$

$a_{86} = 2 + (86-1)(\frac{1}{2}) = \boxed{\frac{89}{2}}$

$a_n = a_1 + (n-1)d$

eg) $a_1 = 5$, $a_n = a_{n-1} + 3$

Examples: Find the first term and common difference of the arithmetic sequence described. Give a recursive formula for the sequence, and write a formula for the *n*th term.

a) 4th term is 3, 20th term is 35

$35 - 3 = 32$ ← total change between 4th & 20th terms

$20 - 4 = 16$ ← how many terms

$\boxed{d = \frac{32}{16} = 2}$ ← go up 2 per term

4th term = 3: (n=4) $a_n = a_1 + (n-1)d$ ← To get to 4th term, add d 3 times. $n-1 = 4-1 = 3$

$3 = a_1 + 3(2)$

$3 = a_1 + 6$

$\boxed{a_1 = -3}$

recursive formula: $\boxed{a_1 = -3;\ a_n = a_{n-1} + 2}$

$a_n = -3 + (n-1)(2)$

$a_n = -3 + 2n - 2$

$\boxed{a_n = 2n - 5}$

b) 5th term is 30, 13th term is –2

$-2 - 30 = -32$

$13 - 5 = 8$

$\boxed{d = \frac{-32}{8} = -4}$

5th term = 30: (n=5) $30 = a_1 + 4(-4)$ ← To get to 5th term, add d 4 times. $n-1 = 5-1 = 4$

$30 = a_1 - 16$

$\boxed{a_1 = 46}$

recursive formula: $\boxed{a_1 = 46;\ a_n = a_{n-1} - 4}$

$a_n = 46 + (n-1)(-4)$

$a_n = 46 - 4n + 4$

$\boxed{a_n = -4n + 50}$

Sum of an Arithmetic Sequence

The sum S_n of the first n terms of an arithmetic sequence $\{a_n\}$ with first term a_1 and common difference d is given by $S_n = a_1 + a_2 + a_3 + \ldots + a_n = \frac{n}{2}(a_1 + a_n)$.

Examples: Find each sum.

a) $-1 + 3 + 7 + \ldots + (4n - 5)$

$S_n = \frac{n}{2}(-1 + 4n - 5)$

$= \frac{n}{2}(4n - 6) = \frac{4n^2 - 6n}{2}$

$= \boxed{2n^2 - 3n}$

b) $1 + 3 + 5 + \ldots + 59$

Need to figure out how many #s are being added (n).

$a_1 = 1$, $a_n = 59$, $d = 2$

$a_n = a_1 + (n-1)d$

$59 = 1 + (n-1)2$

$\frac{58}{2} = \frac{2(n-1)}{2}$

$n - 1 = 29$

$n = 30$

$S_{30} = \frac{30}{2}(1 + 59)$

$= 15(60) = \boxed{900}$

c) $7 + 1 - 5 - 11 - \ldots - 299$

$a_1 = 7$, $a_n = -299$, $d = -6$

$-299 = 7 + (n-1)(-6)$

$\frac{-306}{-6} = \frac{-6(n-1)}{-6}$

$51 = n - 1$

$n = 52$

$S_{52} = \frac{52}{2}(7 + (-299))$

$= \boxed{-7592}$

d) $\sum_{k=1}^{90}(3 - 2k)$ ← n → 90; Arithmetic

$n = 90$

$a_1 = 3 - 2(1) = 1$

$a_{90} = 3 - 2(90) = -177$

Can use $S_n = \frac{n}{2}(a_1 + a_n)$ instead of summation formulas from previous section. Faster!

$\frac{90}{2}(1 + (-177)) = \boxed{-7920}$

e) $\sum_{k=1}^{80}\left(\frac{k}{3} + \frac{1}{2}\right)$

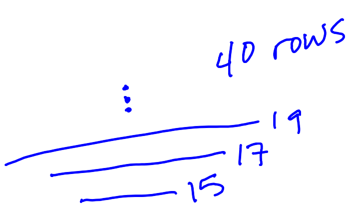
$n = 80$

$a_1 = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$

$a_{80} = \frac{80}{3} + \frac{1}{2} = \frac{163}{6}$

$\frac{80}{2}\left(\frac{5}{6} + \frac{163}{6}\right) = \boxed{1120}$

Example: The corner section of a football stadium has 15 seats in the first row and 40 rows in all. Each successive row contains two additional seats. How many seats are in this section?



$a_1 = 15$

$n = 40$

$d = 2$

$S_{40} = \frac{n}{2}(a_1 + a_n)$

First need to know how many seats in row 40

$a_{40} = 15 + (40-1)(2)$

$= 93$

$S_{40} = \frac{40}{2}(15 + 93)$

$= \boxed{2160 \text{ seats}}$