Sequences

A ***sequence*** is a function whose domain is the set of positive integers.

If the domain of $f(x) = 3x - 2$ is limited to positive integers, the sequence 1, 4, 7, 10, 13… is generated.

We write the above sequence using ***sequence notation*** as $\{a_n\} = \{3n - 2\}$. Here, a is a term in the sequence, and n is the term number. For example, the tenth term in this sequence would be given by $a_{10} = 3(10) - 2 = 28$.

Examples: Write the first five terms of each sequence:

a) $\{s_n\} = \{n^2 + 1\}$

$s_1 = 1^2 + 1 = 2$

$s_2 = 2^2 + 1 = 5$

$s_3 = 3^2 + 1 = 10$

$s_4 = 4^2 + 1 = 17$

$s_5 = 5^2 + 1 = 26$

b) $\{c_n\} = \{(-1)^{n+1} \cdot n^2\}$

$c_1 = (-1)^{1+1} \cdot 1^2 = 1$

$c_2 = (-1)^{2+1} \cdot 2^2 = -4$

$c_3 = (-1)^{3+1} \cdot 3^2 = 9$

$c_4 = (-1)^{4+1} \cdot 4^2 = -16$

$c_5 = (-1)^{5+1} \cdot 5^2 = 25$

c) $\{t_n\} = \left\{ \dfrac{(-1)^n}{(n+1)(n+2)} \right\}$

$t_1 = \dfrac{(-1)^1}{(1+1)(1+2)} = \dfrac{-1}{6}$

$t_2 = \dfrac{(-1)^2}{(2+1)(2+2)} = \dfrac{1}{12}$

$t_3 = \dfrac{(-1)^3}{(3+1)(3+2)} = \dfrac{-1}{20}$

$t_4 = \dfrac{(-1)^4}{(4+1)(4+2)} = \dfrac{1}{30}$

$t_5 = \dfrac{(-1)^5}{(5+1)(5+2)} = \dfrac{-1}{42}$

Note: The term $(-1)^n$ makes the odd terms of a sequence negative and the even terms positive. The terms $(-1)^{n+1}$ or $(-1)^{n-1}$ make the even terms negative and the odd terms positive.

Examples: Write down the nth term of the sequence suggested by each pattern.

a) $-\dfrac{2}{3}, \dfrac{4}{9}, -\dfrac{8}{27}, \dfrac{16}{81}, \ldots$

n	a_n
1	$-\frac{2}{3}$
2	$\frac{4}{9}$
3	$-\frac{8}{27}$
4	$\frac{16}{81}$
n	?

negatives alternate
tops multiply by 2
bottoms multiply by 3

$\{a_n\} = \left\{ (-1)^n \dfrac{2^n}{3^n} \right\}$

or $\left\{ \left(-\dfrac{2}{3}\right)^n \right\}$

b) $1, \dfrac{1}{2}, 3, \dfrac{1}{4}, 5, \dfrac{1}{6}, 7, \dfrac{1}{8}, \ldots$

n	a_n
1	1
2	$\frac{1}{2}$
3	3
4	$\frac{1}{4}$
5	5
6	$\frac{1}{6}$
⋮	⋮

odd terms = term #
even terms = $\dfrac{1}{\text{term \#}}$

$\{a_n\} = \left\{ \begin{array}{ll} n & \text{if } n \text{ is odd} \\ \frac{1}{n} & \text{if } n \text{ is even} \end{array} \right\}$

The Factorial Symbol

If $n \geq 0$ is an integer, then the symbol $n!$ is defined as follows:

$0! = 1 \qquad 1! = 1 \qquad 2! = 2 \cdot 1 = 2 \qquad 3! = 3 \cdot 2 \cdot 1 = 6 \qquad 4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$

$n! = n \cdot (n-1) \cdot \ldots \cdot 3 \cdot 2 \cdot 1 \qquad\qquad n! = n \cdot (n-1)!$

Examples: Evaluate the following:

a) $6!$

$6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

$= \boxed{720}$

b) $\dfrac{7!}{4!} = \dfrac{7 \cdot 6 \cdot 5 \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}$

$= \boxed{210}$

c) $\dfrac{8!}{6!2!} = \dfrac{8 \cdot 7 \cdot \cancel{6!}}{\cancel{6!} \cdot 2 \cdot 1}$

$= \boxed{28}$

1 1 2 3 5 8 13 21 34 55 89 ...

$a_1 = 1 \quad a_2 = 1 \quad a_n = a_{n-2} + a_{n-1}$

Recursive Formula: A recursively-defined sequence assigns a value to the first term or first few terms, then specifies the nth term by a formula or equation that involves one or more of the previous terms.

a_n = nth term $\quad a_{n-1}$ = previous term $\quad n$ = term #

Examples: Write the first five terms of each recursively-defined sequence.

a) $a_1 = 3,\ a_n = 4 - a_{n-1}$

$n=2: \quad a_2 = 4 - a_1 = 4 - 3 = 1$

$n=3: \quad a_3 = 4 - a_2 = 4 - 1 = 3$

$n=4: \quad a_4 = 4 - a_3 = 4 - 3 = 1$

$n=5: \quad a_5 = 4 - a_4 = 4 - 1 = 3$

b) $a_1 = -2,\ a_n = n + 3a_{n-1}$

$a_2 = 2 + 3a_1 = 2 + 3(-2) = -4$

$a_3 = 3 + 3a_2 = 3 + 3(-4) = -9$

$a_4 = 4 + 3a_3 = 4 + 3(-9) = -23$

$a_5 = 5 + 3a_4 = 5 + 3(-23) = -64$

c) $a_1 = -1,\ a_2 = 1,\ a_n = a_{n-2} + na_{n-1}$

$a_3 = a_1 + 3a_2 = -1 + 3(1) = 2$

$a_4 = a_2 + 4a_3 = 1 + 4(2) = 9$

$a_5 = a_3 + 5a_4 = 2 + 5(9) = 47$

d) $a_1 = A,\ a_n = ra_{n-1}$

$a_2 = ra_1 = rA$

$a_3 = ra_2 = r(rA) = r^2A$

$a_4 = ra_3 = r(r^2A) = r^3A$

$a_5 = ra_4 = r(r^3A) = r^4A$

Summation Notation

The symbol Σ (the Greek letter sigma) is an instruction to add up the terms of a sequence. The integer k is called the ***index*** of the sum; it tells you where to start the sum and where to end it. The expression $\sum_{k=1}^{n} a_k$ is an instruction to add the terms a_k of the sequence $\{a_n\}$ starting with $k=1$ and ending with $k=n$. It is read, "the sum of a_k from $k=1$ to $k=n$."

n ← where to stop; a_k ← formula for sequence; Σ ← add up all the terms; $k=1$ ← where to start

Examples: Express each sum using summation notation.

a) $1 + 3 + 5 + 7 + \ldots + [2(12) - 1]$

$2(1)-1 \quad 2(2)-1 \quad 2(3)-1 \quad 2k-1$

$\sum_{k=1}^{12} (2k-1)$

b) $\frac{1}{e} + \frac{2}{e^2} + \frac{3}{e^3} + \cdots + \frac{k}{e^k} + \frac{n}{e^n}$

$\boxed{\sum_{k=1}^{n} \frac{k}{e^k}}$

Examples: Write out each sum.

a) $\sum_{k=1}^{n} (k+1)^2$

$= (1+1)^2 + (2+1)^2 + (3+1)^2 + \ldots + (n+1)^2$

$= 2^2 + 3^2 + 4^2 + \ldots + (n+1)^2$

$= \boxed{4 + 9 + 16 + \ldots + (n+1)^2}$

b) $\sum_{k=0}^{n-1} \left(\frac{1}{3^{k+1}}\right) = \frac{1}{3^{0+1}} + \frac{1}{3^{1+1}} + \frac{1}{3^{2+1}} + \ldots + \frac{1}{3^{n-1+1}}$

$= \frac{1}{3^1} + \frac{1}{3^2} + \frac{1}{3^3} + \ldots + \frac{1}{3^n}$

$= \boxed{\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots + \frac{1}{3^n}}$

Properties of Sequences

If $\{a_n\}$ and $\{b_n\}$ are two sequences, and c is a real number, then:

- $\sum_{k=1}^{n} (ca_k) = ca_1 + ca_2 + \ldots + ca_n = c(a_1 + a_2 + \ldots + a_n) = c\sum_{k=1}^{n} a_k$
- $\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$
- $\sum_{k=1}^{n} (a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$
- $\sum_{k=j}^{n} a_k = \sum_{k=1}^{n} a_k - \sum_{k=1}^{j-1} a_k$, where $1 < j < n$.

$\sum_{k=1}^{3} 5k = 5\cdot1 + 5\cdot2 + 5\cdot3 = 5(1+2+3)$

$= 5\sum_{k=1}^{3} k$

Formulas for Sums of Sequences

- $\sum_{k=1}^{n} c = \underbrace{c+c+\ldots+c}_{n\text{ times}} = cn$ $\quad \sum_{k=1}^{5} 2 = 2+2+2+2+2 = 2(5) = 10$
- $\sum_{k=1}^{n} k = 1+2+3+\ldots+n = \frac{n(n+1)}{2}$ $\quad \sum_{k=1}^{10} k = 1+2+3+4+5+6+7+8+9+10 = \frac{10(10+1)}{2} = 55$
- $\sum_{k=1}^{n} k^2 = 1^2+2^2+3^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6}$ $\quad \sum_{k=1}^{4} k^2 = 1^2+2^2+3^2+4^2 = \frac{4(4+1)(2(4)+1)}{6} = 30$
- $\sum_{k=1}^{n} k^3 = 1^3+2^3+3^3+\ldots+n^3 = \left[\frac{n(n+1)}{2}\right]^2$ $\quad \sum_{k=1}^{4} k^3 = 1^3+2^3+3^3+4^3 = \left[\frac{4(4+1)}{2}\right]^2 = 100$

Examples: Find the sum of each sequence.

a) $\sum_{k=1}^{5}(5k) = 5\sum_{k=1}^{5} k$

$= 5\left(\frac{5(5+1)}{2}\right)$

$= 5(15) = \boxed{75}$

b) $\sum_{k=1}^{26}(3k-7) = 3\sum_{k=1}^{26} k - \sum_{k=1}^{26} 7$

$= 3\left(\frac{26(26+1)}{2}\right) - 7(26)$

$= 3(351) - 182 = \boxed{871}$

c) $\sum_{k=1}^{14}(k^2-4) = \sum_{k=1}^{14} k^2 - \sum_{k=1}^{14} 4$

$= \frac{14(14+1)(2(14)+1)}{6} - 14(4)$

$= 1015 - 56 = \boxed{959}$

d) $\sum_{k=8}^{40}(-3k) = -3\sum_{k=8}^{40} k = -3\left(\sum_{k=1}^{40} k - \sum_{k=1}^{7} k\right)$

8th through 40th terms = all 40 terms − 1st 7 terms

$= -3\left(\frac{40(40+1)}{2} - \frac{7(7+1)}{2}\right)$

$= -3(820-28)$

$= \boxed{-2376}$

e) $\sum_{k=4}^{24} k^3 = \sum_{k=1}^{24} k^3 - \sum_{k=1}^{3} k^3$

4th through 24th terms = all 24 terms − 1st 3 terms

$\left[\frac{24(24+1)}{2}\right]^2 - \left[\frac{3(3+1)}{2}\right]^2$

$300^2 - 6^2 = \boxed{89{,}964}$