Sequences

A ***sequence*** is a function whose domain is the set of positive integers.

If the domain of $f(x) = 3x - 2$ is limited to positive integers, the sequence 1, 4, 7, 10, 13… is generated.

We write the above sequence using ***sequence notation*** as $\{a_n\} = \{3n - 2\}$. Here, a is a term in the sequence, and n is the term number. For example, the tenth term in this sequence would be given by $a_{10} = 3(10) - 2 = 28$.

Examples: Write the first five terms of each sequence:

a) $\{s_n\} = \{n^2 + 1\}$ b) $\{c_n\} = \{(-1)^{n+1} \cdot n^2\}$ c) $\{t_n\} = \left\{\dfrac{(-1)^n}{(n+1)(n+2)}\right\}$

Note: The term $(-1)^n$ makes the odd terms of a sequence negative and the even terms positive. The terms $(-1)^{n+1}$ or $(-1)^{n-1}$ make the even terms negative and the odd terms positive.

Examples: Write down the nth term of the sequence suggested by each pattern.

a) $-\frac{2}{3}, \frac{4}{9}, -\frac{8}{27}, \frac{16}{81}, \ldots$ b) $1, \frac{1}{2}, 3, \frac{1}{4}, 5, \frac{1}{6}, 7, \frac{1}{8}, \ldots$

The Factorial Symbol

If $n \geq 0$ is an integer, then the symbol $n!$ is defined as follows:

$0! = 1$ $1! = 1$ $2! = 2 \cdot 1 = 2$ $3! = 3 \cdot 2 \cdot 1 = 6$ $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$

$n! = n \cdot (n-1) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$ $n! = n \cdot (n-1)!$

Examples: Evaluate the following:

a) $6!$ b) $\dfrac{7!}{4!}$ c) $\dfrac{8!}{6!2!}$

***Recursive Formula*:** A recursively-defined sequence assigns a value to the first term or first few terms, then specifies the *n*th term by a formula or equation that involves one or more of the previous terms.

Examples: Write the first five terms of each recursively-defined sequence.

a) $a_1 = 3,\ a_n = 4 - a_{n-1}$

b) $a_1 = -2,\ a_n = n + 3a_{n-1}$

c) $a_1 = -1,\ a_2 = 1,\ a_n = a_{n-2} + na_{n-1}$

d) $a_1 = A,\ a_n = ra_{n-1}$

Summation Notation

The symbol Σ (the Greek letter sigma) is an instruction to add up the terms of a sequence. The integer k is called the ***index*** of the sum; it tells you where to start the sum and where to end it. The expression $\sum_{k=1}^{n} a_k$ is an instruction to add the terms a_k of the sequence $\{a_n\}$ starting with $k = 1$ and ending with $k = n.$ It is read, "the sum of a_k from $k = 1$ to $k = n.$"

Examples: Express each sum using summation notation.

a) $1+3+5+7+...+\left[2(12)-1\right]$

b) $\frac{1}{e}+\frac{2}{e^2}+\frac{3}{e^3}+\cdots+\frac{n}{e^n}$

Examples: Write out each sum.

a) $\sum_{k=1}^{n}(k+1)^2$

b) $\sum_{k=0}^{n-1}\left(\frac{1}{3^{k+1}}\right)$

Properties of Sequences

If $\{a_n\}$ and $\{b_n\}$ are two sequences, and *c* is a real number, then:

- $\sum_{k=1}^{n}(ca_k) = ca_1 + ca_2 + ... + ca_n = c(a_1 + a_2 + ... + a_n) = c\sum_{k=1}^{n} a_k$
- $\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$
- $\sum_{k=1}^{n}(a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$
- $\sum_{k=j}^{n} a_k = \sum_{k=1}^{n} a_k - \sum_{k=1}^{j-1} a_k$, where $1 < j < n.$

Formulas for Sums of Sequences

- $\sum_{k=1}^{n} c = \underbrace{c + c + \ldots + c}_{n \text{ times}} = cn$
- $\sum_{k=1}^{n} k = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$
- $\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$
- $\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \ldots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$

Examples: Find the sum of each sequence.

a) $\sum_{k=1}^{5} (5k) =$

b) $\sum_{k=1}^{26} (3k - 7) =$

c) $\sum_{k=1}^{14} (k^2 - 4) =$

d) $\sum_{k=8}^{40} (-3k) =$

e) $\sum_{k=4}^{24} k^3 =$