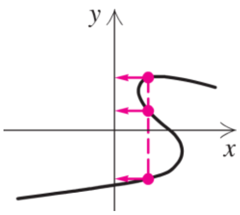
Graphs of Functions

Not every collection of points in the *xy*-plane represents a function. Remember, for a function, each number *x* in the domain has exactly one image *y* in the range. The graph of the function must satisfy the *vertical line test*.

Vertical Line Test



If it is possible for a vertical line to cross a graph more than once, then the graph is not the graph of a function.

The graph at left is not a function because three *y*-values correspond to one *x*-value.

Domain: The set of all inputs (the *x*-values) of a relation.

- If a relation is represented by a graph, the domain is the set of the *x*-coordinates of all points on the graph. You can think of it as the graph's shadow on the *x*-axis.

Range: The set of all outputs (the *y*-values) of a relation.

- If a relation is represented by a graph, the range is the set of the *y*-coordinates of all points on the graph. You can think of it as the graph's shadow on the *y*-axis.

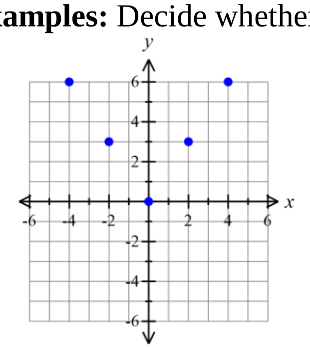
If the graph is a set of unconnected points, the domain and range are simply lists of the *x* and *y* coordinates, respectively. However, if the graphs are continuous, they contain an infinite number of points, so it becomes impossible to list the coordinates. One way we solve this is to use ***interval notation.***

Interval Notation

- Write the numbers where the interval starts and stops, separated by a comma.
- If the number is *included*, put it in a bracket [or].
- If the number is *not included* (asymptote or open circle), put it in parentheses (or).
- If the graph extends forever in a particular direction, use $-\infty$ or ∞. These always get put in parentheses.
- If there are multiple intervals, they are connected with a $\cup$ sign.

Examples: Decide whether each graph represents a function. Then find the domain and range.

a)



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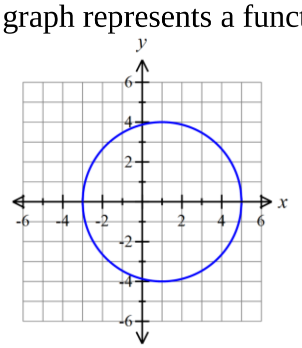
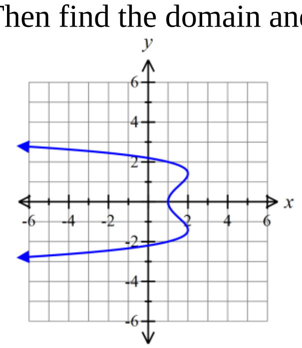
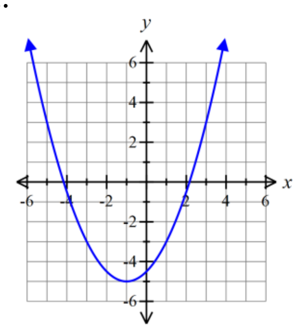


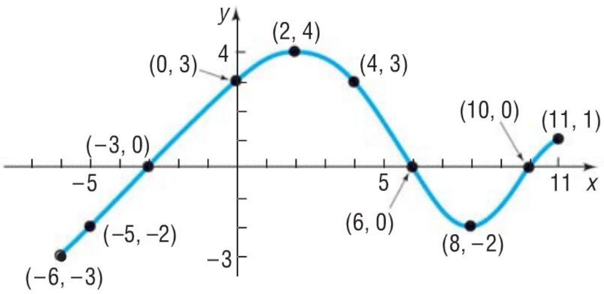
c)



d)



Example:



a) Find $f(0)$ and $f(-6)$.

b) Find $f(6)$ and $f(11)$.

c) Is $f(3)$ positive or negative?

d) Is $f(-4)$ positive or negative?

e) For what values of *x* is $f(x) = 0$?

f) For what values of *x* is $f(x) > 0$?

g) What is the domain of f?

h) What is the range of f?

i) What are the *x*-intercepts?

j) What is the *y*-intercept?

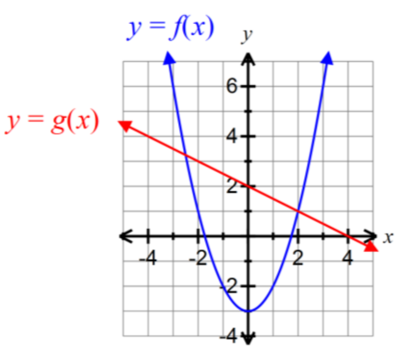
k) How many times does the line $y = 1/2$ intersect the graph?

l) How many times does the line $x = 5$ intersect the graph?

m) For what values of *x* does $f(x) = 3$?

n) For what values of *x* does $f(x) = -2$?

Example:



a) Find $(f+g)(0)$

b) Find $(f-g)(-2)$

c) Find $(f \cdot g)(3)$

d) Find $\left(\frac{f}{g}\right)(2)$

Intercepts

x-Intercepts: The points where a graph crosses the *x*-axis. They have the form $(x, 0)$.

- To find the *x*-intercept(s), set y or $f(x) = 0$ and solve for x.

y-Intercepts: The points where a graph crosses the *y*-axis. They have the form $(0, y)$.

- To find the *y*-intercept(s), set $x = 0$ and solve for y.

Example: Consider the function $f(x) = \dfrac{x^2 + 2}{x + 4}$.

a) Is the point $\left(-1, \dfrac{3}{5}\right)$ on the graph of f?

b) If $x = 0$, what is $f(x)$? What point is on the graph of f?

c) If $f(x) = 3$, what is *x*? What point(s) are on the graph of f?

d) What is the domain of f?

e) Find the *x*-intercepts, if any, of the graph of f.

f) Find the *y*-intercept, if there is one, of the graph of f.

Example: A golf ball is hit with an initial velocity of 130 feet per second at an inclination of 45° to the horizontal. In physics, it is established that the height, *h*, of the golf ball, in feet, is given by the function $h(x) = \dfrac{-32x^2}{130^2} + x$, where *x* is the horizontal distance that the golf ball has traveled, in feet.

a) Determine the height of the golf ball after it has traveled 100 feet, 300 feet, and 500 feet.

b) How far forward has the golf ball traveled when it lands?

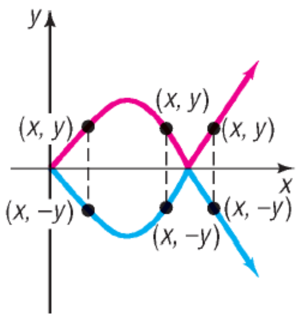
c) Graph the path of the ball using a graphing calculator and determine how far the ball travels before it reaches its maximum height and what its maximum height is.

Symmetry

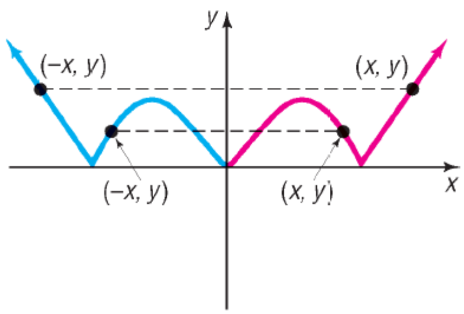
1. A graph is said to be ***symmetric with respect to the x-axis*** if, for every point (x, y) on the graph, the point $(x, -y)$ is also on the graph.

2. A graph is said to be ***symmetric with respect to the y-axis*** if, for every point (x, y) on the graph, the point $(-x, y)$ is also on the graph.

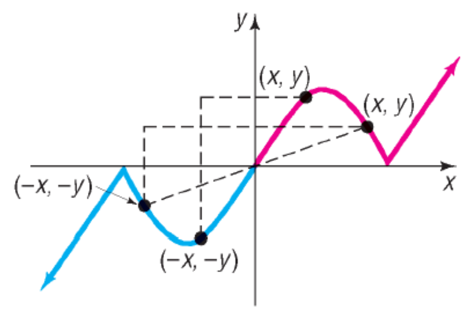
3. A graph is said to be ***symmetric with respect to the origin*** if, for every point (x, y) on the graph, the point $(-x, -y)$ is also on the graph.



Symmetry with respect to the *x*-axis



Symmetry with respect to the *y*-axis



Symmetry with respect to the origin

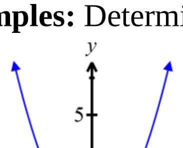
Even and Odd Functions: The words **even** and **odd**, when applied to a function f, describe the symmetry that exists for the graph of the function.

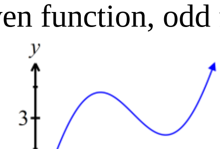
Even Function: A function f is even if, for every number x in its domain, the number $-x$ is also in the domain and $f(-x) = f(x)$. Even functions are **symmetric with respect to the *y*-axis.**

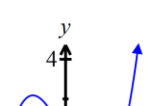
Odd Function: A function f is odd if, for every number x in its domain, the number $-x$ is also in the domain and $f(-x) = -f(x)$. Odd functions are **symmetric with respect to the origin.**

Question: Why is there no term for a function that is symmetric with respect to the *x*-axis?

Examples: Determine whether each graph is an even function, odd function, or neither.







a) b) c)

Testing whether a function is even, odd, or neither:

1. Find $f(-x)$. If $f(-x) = f(x)$, the function is even. If not, continue.
2. Find $-f(x)$. If $f(-x) = -f(x)$, the function is odd.

Examples: Determine algebraically whether each of the following functions is even, odd, or neither. Then determine whether the graph is symmetric with respect to the *y*-axis, the origin, or neither.

a) $f(x) = x^2 - 2$

b) $f(x) = x^3 + x$

c) $f(x) = 4x^3 + x^2 - 1$

d) $f(x) = |x| + 5$

e) $f(x) = \dfrac{2x}{x^2 - 5}$

f) $f(x) = \sqrt[3]{x} + 7\sqrt{x}$

Increasing, Decreasing, and Constant Graphs: If you look from left to right along the graph of the function, you will notice parts are *rising*, parts are *falling* and parts are *horizontal*. In such cases, the function is described as *increasing*, *decreasing*, or *constant*, respectively.

Definitions:

- A function f is ***increasing*** on an interval if for any choice of x_1 and x_2 in the interval, where $x_1 < x_2$, then $f(x_1) < f(x_2)$.
- A function f is ***decreasing*** on an interval if for any choice of x_1 and x_2 in the interval, where $x_1 < x_2$,

then $f(x_1) > f(x_2)$.

- A function f is ***constant*** on an interval if for any choice of x_1 and x_2 in the interval, where $x_1 < x_2$, then $f(x_1) = f(x_2)$.

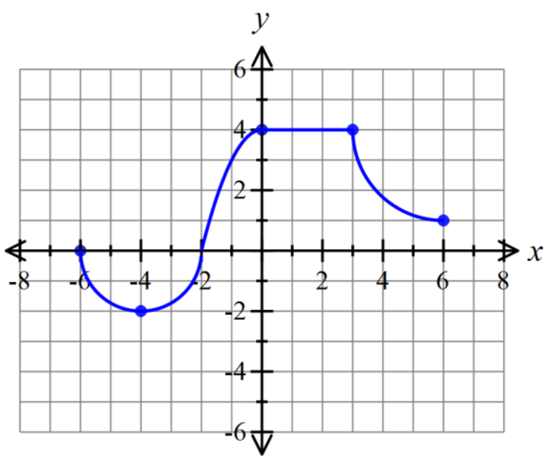
Increasing **Decreasing** **Constant**

★ When you are asked to state where the graph is increasing, decreasing, and constant, write the intervals of ***x*-coordinates** from **left to right.**

★ Always use ( ) for increasing, decreasing, and constant. Never use [ ].

Example: Determine where each graph is increasing, decreasing, and constant.

a)



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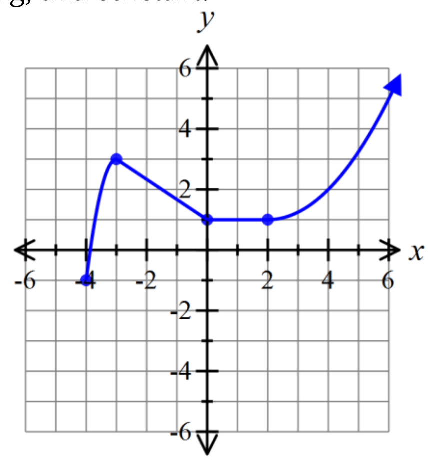


Maxima and Minima

When a point is higher that all the points near it, it is called a ***relative*** or ***local maximum*.** When a point is lower than all the points near it, it is called a ***relative*** or ***local minimum*.** The highest and lowest points on the entire graph are called the ***absolute maximum*** and ***absolute miniumum.***

★ Absolute maxima and minima are also considered to be relative maxima and minima.

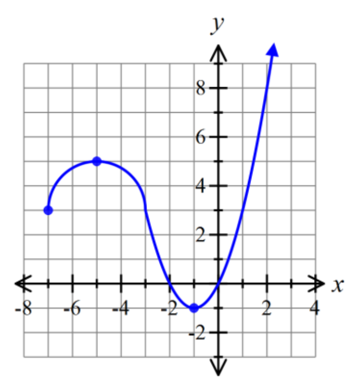
- If you are asked for a ***maximum point or a minimum point,*** write the answer as an ordered pair.
- If you are asked for a ***maximum value or a minimum value,*** the answer is the ***y-coordinate.***

★ **Note:** If a question asks "Where…", "On what interval(s)…", or "At what number(s)…", it is asking for *x*-coordinates. If it asks "What is..." or "Find the value of…", it is asking for a *y*-coordinate.

Example:

a) At what number(s), if any, does f have a local maximum?

b) What are the local maximum value(s)?



c) At what number(s), if any, does f have a relative minimum?

d) What are the relative minimum value(s)?

e) Does f have an absolute maximum? If so, where is it? What is the absolute maximum value?

f) Does f have an absolute minimum? If so, where is it? What is the absolute minimum value?